

# On the extreme values of the Riemann zeta function on random intervals of the critical line

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## Abstract

In the present paper, we show that under the Riemann hypothesis, and for fixed  $h, \epsilon > 0$ , the supremum of the real and the imaginary parts of  $\log \zeta(1/2 + it)$  for  $t \in [UT - h, UT + h]$  are in the interval  $[(1 - \epsilon) \log \log T, (1 + \epsilon) \log \log T]$  with probability tending to 1 when  $T$  goes to infinity, if  $U$  is uniformly distributed in  $[0, 1]$ . This proves a weak version of a conjecture by Fyodorov, Hiary and Keating, which has recently been intensively studied in the setting of random matrices. We also unconditionally show that the supremum of  $\Re \log \zeta(1/2 + it)$  is at most  $\log \log T + g(T)$  with probability tending to 1,  $g$  being any function tending to infinity at infinity.

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## 1 Introduction

In relation with the Riemann hypothesis, the behavior of the function  $\zeta$  at or near the critical line has been extensively studied, in particular its maximal order of magnitude. For example, a consequence of the Riemann hypothesis is the Lindelöf hypothesis, which claims that for all  $\alpha > 0$ ,  $|\zeta(1/2 + it)| = \mathcal{O}(t^\alpha)$  if  $t > 3$ : in fact, this result can be improved to  $|\zeta(1/2 + it)| \ll \exp(\mathcal{O}(\log t / \log \log t))$  (see Titchmarsh [Tit86], Theorem 14.14 (A)). If one does not assume the Riemann hypothesis, the best known result in this direction, due to Bourgain [Bou14], is that the estimate is true for all  $\alpha > 13/84$ . On the other hand, it is also known (see

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Titchmarsh [Tit86], p. 209) that for all  $c < 3/4$ ,  $|\zeta(1/2 + it)| \geq \exp(c\sqrt{\log t / \log \log t})$  for infinitely many values of  $t$ .

It has been conjectured by Farmer, Gonek, and Hughes [FGH07], that the supremum of  $|\zeta(1/2 + it)|$  on  $[0, T]$  is  $\exp((1 + o(1))\sqrt{(1/2) \log T \log \log T})$  for  $T \rightarrow \infty$ : their heuristics is related to the fact that the large values of  $|\zeta|$  are expected to be roughly independent when they occur at points which are sufficiently far from each other.

A classical result by Selberg [Sel91] shows that for  $t$  random, uniform on  $[0, T]$ ,  $\log |\zeta(1/2 + it)|/\sqrt{\log \log T}$  converges in law towards a centered Gaussian random variable of variance  $1/2$ , however, it does not give information about the dependence between the values of  $\log |\zeta|$  at different points. It has been proven by Hughes, Nikeghbali and Yor [HNY08] that, in a sense which is made precise in their article, the values of  $\log |\zeta|$  are independent if they are taken at points which are very distant from each other, and in the other hand, Bourgade [Bou10] has proven that the values are correlated if they are taken at points which are close enough to each other. For example, for  $t$  uniform on  $[0, T]$ , and  $a \in [0, 1]$ , we have that  $(\log |\zeta(1/2 + it)|/\sqrt{\log \log T}, \log |\zeta(1/2 + i(t + (\log T)^{-a}))|/\sqrt{\log \log T})$  tends to a two-dimensional centered Gaussian vector, whose correlation is equal to  $a$ . In [FK14] and [FHK12], Fyodorov, Hiary and Keating make heuristic computations suggesting that  $\log |\zeta|$  behaves like a centered Gaussian field, whose correlation between random points in  $[0, T]$  at distance  $u \leq 1$  is given by  $\min(|\log u|, \log \log T)$ . A more sophisticated random field is constructed by Saksman and Webb [SW16], as a limiting random distribution for the function  $\zeta$  itself.

From the comparison between  $\log |\zeta|$  and a log-correlated Gaussian field, and some moment computations coming from techniques used in statistical mechanics, Fyodorov, Hiary and Keating have formulated a conjecture concerning the supremum of  $\log |\zeta|$  on random intervals of fixed length, which can be stated as follows:

**Conjecture 1.1.** *For  $U$  uniform on  $[0, 1]$  and  $h > 0$  fixed, the family of random variables:*

$$\left( \sup_{t \in [UT-h, UT+h]} \log \left| \zeta \left( \frac{1}{2} + it \right) \right| - \left( \log \log T - \frac{3}{4} \log \log \log T \right) \right)_{T > 3} \quad (1.1)$$

*tends to a limiting distribution when  $T$  goes to infinity.*

The limiting distribution is explicitly given, and is related to the sum of two independent Gumbel random variables.

The study of the maximum of log-correlated Gaussian fields has been done in several different and fairly general settings. In the case of branching Brownian motion and branching random walks, for which a tree structure is explicit, the main general results have been proven by Aïdékon, Hu and Shi (see [HS09, AS10, Aïd13]). In the case of log-correlated Gaussian fields on  $[0, 1]^d$  ( $d \geq 1$ ) a limit theorem has been proven by Madaule [Mad15], and then generalized by Ding, Roy and Zeitouni [DRZ15]. The discrete Gaussian Free Fields have been studied by Bramson, Ding and Zeitouni in [BZ12] and [BDZ16].

An analogy between the Riemann zeta function and the characteristic polynomial of random matrices has been developed in the last decades, following the idea by Pólya and Hilbert that there may be a way to see the non-trivial zeros of  $t \mapsto \zeta(1/2 + it)$  as the eigenvalues of some Hermitian operator. In relation with this analogy, we can mention the results by Montgomery [Mon73] on the pair correlation of zeros of  $\zeta$ , which have been observed by Dyson to be similar to what we obtain for eigenvalues of random Hermitian or unitary matrices, the conjecture by Keating and Snaith [KS00] on the moments of  $\zeta$  on the critical line, and the limit theorems by Katz and Sarnak [KS99] on analogs of the Riemann zeta function, constructed from algebraic curves on function fields.

In relation with this analogy, Fyodorov, Hiary and Keating have stated a version of Conjecture 1.1 in the setting of random matrix theory, which says that

$$\left( \sup_{|z|=1} \log |X_n(z)| - \left( \log n - \frac{3}{4} \log \log n \right) \right)_{n \geq 2} \quad (1.2)$$

tends to a limiting distribution (namely, the law of the average of two independent Gumbel variables), if  $X_n$  is the characteristic polynomial of a Haar-distributed,  $n \times n$ , random unitary matrix.

This conjecture is not fully proven, but Chhaibi, Madaule and Najnudel in [CMN16] have recently shown that the family (1.2) of random variables is tight, improving successive results by Arguin, Belius and Bourgade [ABB15] and by Paquette and Zeitouni [PZ16].

Proving Conjecture 1.1, or even only the tightness of (1.1), seems to be much more difficult than the corresponding results in the setting of random matrices. One of the reasons is that there is a priori less randomness in the setting of Riemann zeta function. Indeed, in [ABH15], Arguin, Belius and Harper consider a randomized version of the Riemann zeta function and partially show an analog of Conjecture 1.1.

The main goal of the present paper is to show that one can progress towards Conjecture 1.1 in the original setting of the Riemann zeta function, without extra randomness. More precisely, we will prove that under the Riemann hypothesis, the leading order term in Conjecture 1.1 is the correct one, which correspond to the precision of the result by Arguin, Belius and Bourgade [ABB15] for random matrices.

In all this paper,  $\log \zeta$  will be defined as the unique version of the logarithm of zeta which is real on  $(1, \infty)$ , well-defined and continuous everywhere, except on the closed half-lines at the left of the zeros and the pole of  $\zeta$ . The values for which  $\log \zeta$  is not well-defined will be considered to be implicitly excluded from all infima and suprema where they are involved. Note that despite this exclusion, we still have, for all  $a < b$ ,

$$\sup_{t \in [a, b]} \Re \log \zeta \left( \frac{1}{2} + it \right) = \sup_{t \in [a, b]} \log \left| \zeta \left( \frac{1}{2} + it \right) \right|,$$

since  $\log |\zeta|$  is continuous on the critical line except at the zeros of  $\zeta$ , where it is equal to  $-\infty$  and then is not involved in the supremum in the right-hand side. With this convention, our main result is the following:

**Theorem 1.2.** *Let us assume the Riemann hypothesis. Then, if  $h > 0$ ,  $\epsilon > 0$  are fixed, if  $T > 3$  and if  $U$  is a uniform random variable in  $[0, 1]$ , then*

$$\begin{aligned} \mathbb{P} \left( \sup_{t \in [UT-h, UT+h]} \Re \log \zeta \left( \frac{1}{2} + it \right) \in [(1-\epsilon) \log \log T, (1+\epsilon) \log \log T] \right) &\xrightarrow{T \rightarrow \infty} 1, \\ \mathbb{P} \left( \sup_{t \in [UT-h, UT+h]} \Im \log \zeta \left( \frac{1}{2} + it \right) \in [(1-\epsilon) \log \log T, (1+\epsilon) \log \log T] \right) &\xrightarrow{T \rightarrow \infty} 1, \\ \mathbb{P} \left( \inf_{t \in [UT-h, UT+h]} \Im \log \zeta \left( \frac{1}{2} + it \right) \in [-(1+\epsilon) \log \log T, -(1-\epsilon) \log \log T] \right) &\xrightarrow{T \rightarrow \infty} 1. \end{aligned}$$

Unfortunately, we are not able to remove the Riemann hypothesis from the assumptions of this theorem. However, we have an unconditional and stronger result for the upper bound of the real part of  $\log \zeta$ :

**Theorem 1.3.** *Let  $h > 0$ , and let  $g$  be a function from  $[3, \infty)$  to  $\mathbb{R}_+$ , tending to infinity at infinity. Then, for  $T > 3$  and  $U$  uniform on  $[0, 1]$ ,*

$$\mathbb{P} \left( \sup_{t \in [UT-h, UT+h]} \Re \log \zeta \left( \frac{1}{2} + it \right) \leq \log \log T + g(T) \right) \xrightarrow{T \rightarrow \infty} 1.$$

The part of Theorem 1.2 concerning  $\Im \log \zeta$  gives some information on the fluctuations of the distribution of the zeros of  $\zeta$ . From Titchmarsh [Tit86], Theorem 9.3., we deduce the following:

**Corollary 1.4.** *Assume the Riemann hypothesis. For  $t > 0$ , let  $N(t)$  be the number of non-trivial zeros of  $\zeta$  whose imaginary part is in  $[0, t]$ , and let*

$$\Delta(t) := N(t) - \frac{t \log t}{2\pi} + \frac{t(1 + \log(2\pi))}{2\pi}.$$

*Then, if  $h > 0$ ,  $\epsilon > 0$  are fixed, if  $T > 3$  and if  $U$  is a uniform random variable in  $[0, 1]$ ,*

$$\mathbb{P} \left( UT - h > 0, \sup_{t \in [UT-h, UT+h]} \Delta(t) \in \left[ \frac{1-\epsilon}{\pi} \log \log T, \frac{1+\epsilon}{\pi} \log \log T \right] \right) \xrightarrow{T \rightarrow \infty} 1,$$

$$\mathbb{P} \left( UT - h > 0, \inf_{t \in [UT-h, UT+h]} \Delta(t) \in \left[ -\frac{1+\epsilon}{\pi} \log \log T, -\frac{1-\epsilon}{\pi} \log \log T \right] \right) \xrightarrow{T \rightarrow \infty} 1.$$

The proof of our main theorem is divided into several parts.

The upper bound for the real part is covered by Theorem 1.3, which is proven by showing that the supremum of  $|\zeta|$  in the segment  $[1/2 + i(UT - h), 1/2 + i(UT + h)]$  is well-controlled by the values of  $|\zeta|$  at about  $\log T$  points of the segment. Then, we conclude by using classical estimates of the second moment of  $|\zeta|$  on the critical line.

For the upper bound of the imaginary part, we prove that the supremum of  $\Im \log \zeta$  on the same segment is controlled by the supremum of some averages of  $\Im \log \zeta$  around about  $\log T$  points. Then, we show that these averages are close to finite sums indexed by prime numbers, for which we give suitable bounds on the tail of their distribution.

For the lower bound, we use the fact that averaging  $\Re \log \zeta$  or  $\Im \log \zeta$  essentially decreases its supremum, up to some error terms which can be controlled. We then deduce that it is sufficient to get lower bounds on some sums indexed by primes which are explicitly given. These sums are proven to be sufficiently close, in the sense of their Fourier transform, to Gaussian variables with the same covariance structure. After cutting the sum into smaller pieces in order to get some approximative branching structure, we obtain a way to apply the second moment method, as for the lower bound on branching random walks.

The sequel of the paper is structured as follows. In Section 2, we give a proof of Theorem 1.3. In Section 3, we show that if we average the logarithm of the Riemann zeta function around points of the critical line in a suitable way, then we get something close to a finite sum indexed by prime numbers which is explicitly given. In Section 4, we prove the upper bound part of Theorem 1.2. In Section 5, we use the results in Section 3 in order to bound from below the supremum of  $\Re \log \zeta$  or  $\Im \log \zeta$  in terms on a sum on prime numbers which is more tractable than the one given in Section 3. In Section 6, we cut the sum obtained in Section 5 into smaller pieces and show that their joint law is close, in a sense which can be made precise, to a Gaussian family of variables. In Section 7, we finally use the second moment method in order to prove the lower bound part of Theorem 1.2.

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## Notation

In the present paper, we use the Vinogradov notation

$$f \ll g \Leftrightarrow f = \mathcal{O}(g) ,$$

and when the implicit constant depend on parameters (e.g, on a positive real  $h$  and a function  $\varphi$ ), this dependence will be indicated thanks to subscripts (e.g  $\ll_{\varphi,h}$  or  $\mathcal{O}_{\varphi,h}$ ). The Fourier transform is normalized as follows:

$$\widehat{\varphi}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} \varphi(x) dx.$$

Finally,  $\mathcal{P} = \{2, 3, 5, 7, \dots\}$  denotes the set of prime numbers.

## 2 Proof of Theorem 1.3

In [CMN16], in order to get an upper bound for the maximal modulus of the characteristic polynomial, we show that the maximal modulus, on the unit circle, of any polynomial of degree  $n$ , is at most 14 times the maximal modulus on the  $2n$ -th roots of unity (14 is not optimal). In this section, we show a quite similar result, except that the unit circle is replaced by the real line, and the polynomials of a given degree are replaced by the functions whose Fourier transform is supported by a given compact set. The following lemma shows that such functions are uniquely determined and well-controlled by their values on a discrete set of points:

**Lemma 2.1.** *There exists a universal function  $\varphi$  from  $\mathbb{R}$  to  $\mathbb{R}$ , continuous and decaying faster than any power at infinity, satisfying the following property: if  $f$  be a bounded, continuous function whose Fourier transform has a support in  $[-\lambda, \lambda]$  ( $\lambda > 0$ ) in the sense of the distribution, then for all  $x \in \mathbb{R}$ ,*

$$f(x) = \sum_{k \in \mathbb{Z}} \varphi((2\lambda x/\pi) - k) f(k\pi/2\lambda).$$

*Proof.* We take for  $\varphi$  a function whose Fourier transform is real and even (then  $\varphi$  is real-valued), equal to 1 in a neighborhood of  $[-\pi/2, \pi/2]$  and to 0 in a neighborhood of  $\mathbb{R} \setminus (-\pi, \pi)$ , smooth (hence  $\varphi$  is rapidly decaying). In the sense of the distributions, the right-hand side can be written as

$$x \mapsto (\varphi \star (Dg))(2\lambda x/\pi),$$

where  $g$  is the function  $x \mapsto f(x\pi/2\lambda)$  and  $D$  is the sum of Dirac masses at integers. Taking the Fourier transform, we get

$$\mu \mapsto \frac{\pi}{2\lambda} \widehat{\varphi}(\pi\mu/2\lambda) \frac{1}{2\pi} (\widehat{D} \star \widehat{g})(\pi\mu/2\lambda).$$

Now,  $\widehat{D}$  is  $2\pi$  times a sum of Dirac masses at multiples of  $2\pi$ . Hence,

$$\widehat{D} \star \widehat{g}(\pi\mu/2\lambda) = 2\pi \sum_{k \in \mathbb{Z}} \widehat{g}((\pi\mu/2\lambda) + 2k\pi) = 4\lambda \sum_{k \in \mathbb{Z}} \widehat{f}(\mu + 4k\lambda)$$

and the Fourier transform of the right-hand side in the lemma is

$$\mu \mapsto \widehat{\varphi}(\pi\mu/2\lambda) \sum_{k \in \mathbb{Z}} \widehat{f}(\mu + 4k\lambda).$$

In a neighborhood of  $\mathbb{R} \setminus (-2\lambda, 2\lambda)$ , this distribution vanishes because of the choice of  $\varphi$ . For  $\mu \in [-2\lambda, 2\lambda]$ ,  $\mu + 4k\lambda$  can only be in the support of  $\widehat{f}$  for  $k = 0$ . Hence, the distribution is equal to

$$\mu \mapsto \widehat{\varphi}(\pi\mu/2\lambda)\widehat{f}(\mu).$$

It is then, as  $\widehat{f}$ , supported in  $[-\lambda, \lambda]$ , and it is in fact equal to  $\widehat{f}$  since  $\widehat{\varphi}(\pi\mu/2\lambda) = 1$  for  $\mu$  in a neighborhood of  $[-\lambda, \lambda]$ .

The two sides of the equality have the same Fourier transform, and then they are equal in the sense of the distributions. Therefore, we are done, since the right-hand side is a continuous function of  $x$ , by dominated convergence.  $\square$

A consequence of the lemma is the following:

**Proposition 2.2.** *For all  $A > 1$ , for any continuous, bounded function  $f$  whose Fourier transform is supported in  $[-\lambda, \lambda]$ , for all  $x_0 \in \mathbb{R}$ ,  $h, R > 0$ ,*

$$\sup_{x \in [x_0 - h, x_0 + h]} |f(x)| \ll_A \left( \sum_{k \in \mathbb{Z}, |k| \leq \lambda h} |f(x_0 + (k\pi/2\lambda))| + \sum_{k \in \mathbb{Z}, |k| \leq R} \frac{|f(x_0 + (k\pi/2\lambda))|}{1 + |k|^A} + \frac{\sup_{\mathbb{R}} |f|}{1 + R^{A-1}} \right).$$

*Proof.* By translating  $f$ , we can assume  $x_0 = 0$ , moreover, it is sufficient to show the result corresponding to  $R \rightarrow \infty$ , i.e.

$$\sup_{x \in [-h, h]} |f(x)| \ll_A \left( \sum_{k \in \mathbb{Z}, |k| \leq \lambda h} |f(k\pi/2\lambda)| + \sum_{k \in \mathbb{Z}} \frac{|f(k\pi/2\lambda)|}{1 + |k|^A} \right).$$

Now, by the lemma,

$$\sup_{x \in [-h, h]} |f(x)| \leq \sum_{k \in \mathbb{Z}} |f(k\pi/2\lambda)| \sup_{y \in [-(2\lambda h/\pi) - k, (2\lambda h/\pi) - k]} |\varphi(y)|.$$

Now, by the assumption on  $\varphi$  made in the lemma, there exists  $K_A > 0$  such that  $\varphi(y) \leq K_A/(1 + |y|^A)$  for all  $y \in \mathbb{R}$ . Hence, for  $|k| \leq \lambda h$ ,

$$\sup_{y \in [-(2\lambda h/\pi) - k, (2\lambda h/\pi) - k]} |\varphi(y)| \leq K_A \leq K_A \left( 1 + \frac{1}{1 + |k|^A} \right),$$

and, for  $|k| \geq \lambda h$ ,

$$|k| - \frac{2\lambda h}{\pi} \geq |k| - \frac{2|k|}{\pi} \geq |k|/3,$$

and then

$$\sup_{y \in [-(2\lambda h/\pi) - k, (2\lambda h/\pi) - k]} |\varphi(y)| \leq \sup_{|y| \geq |k|/3} |\varphi(y)| \leq \frac{K_A}{1 + (|k|/3)^A} \leq \frac{3^A K_A}{1 + |k|^A}.$$

This gives the desired result.  $\square$

The next step of our proof of Theorem 1.3 is to show that locally on the critical line, the Riemann zeta function is not far from being a function whose Fourier transform has compact support.

**Proposition 2.3.** Let  $Z$  be the function from  $\mathbb{R}$  to  $\mathbb{R}$  given by

$$Z(t) = \zeta(1/2 + it)e^{i\theta(t)},$$

for

$$\theta(t) = \Im \log \Gamma(1/4 + it/2) - \frac{t}{2} \log \pi,$$

where we take the continuous version of  $\Im \log \Gamma(1/4 + it/2)$  vanishing at zero. Then for  $t_0 \geq 2$ , and uniformly on  $t \in [t_0 - t_0^{1/4}, t_0 + t_0^{1/4}]$ , we have

$$Z(t) = 2 \sum_{k=1}^{\lfloor \sqrt{t_0/2\pi} \rfloor} \frac{\cos\left(\frac{t}{2} \log\left(\frac{t_0}{2\pi k^2}\right) - \frac{t_0}{2} - \frac{\pi}{8}\right)}{\sqrt{k}} + \mathcal{O}(t_0^{-1/4}).$$

*Proof.* The Riemann-Siegel formula gives (see [Tit86], p. 89):

$$Z(t) = 2 \sum_{k=1}^{\lfloor \sqrt{t/2\pi} \rfloor} \frac{\cos(\theta(t) - t \log k)}{\sqrt{k}} + \mathcal{O}(t^{-1/4})$$

where complex Stirling formula gives the expansion:

$$\theta(t) = \frac{t}{2} \log(t/2\pi) - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}(1/t).$$

Hence,

$$\begin{aligned} \theta(t) &= \frac{t}{2} \left( \log(t_0/2\pi) + \frac{t-t_0}{t_0} + \mathcal{O}\left(\frac{(t-t_0)^2}{t_0^2}\right) \right) - \frac{t_0}{2} - \frac{t-t_0}{2} - \frac{\pi}{8} + \mathcal{O}(1/t_0) \\ &= \frac{t}{2} \log(t_0/2\pi) - \frac{t_0}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1+(t-t_0)^2}{t_0}\right) \\ &= \frac{t}{2} \log(t_0/2\pi) - \frac{t_0}{2} - \frac{\pi}{8} + \mathcal{O}(t_0^{-1/2}). \end{aligned}$$

We deduce

$$\begin{aligned} Z(t) &= 2 \sum_{k=1}^{\lfloor \sqrt{t/2\pi} \rfloor} \frac{\cos\left(\frac{t}{2} \log\left(\frac{t_0}{2\pi k^2}\right) - \frac{t_0}{2} - \frac{\pi}{8}\right)}{\sqrt{k}} + \mathcal{O}\left(t_0^{-1/2} \sum_{k=1}^{\lfloor \sqrt{t/2\pi} \rfloor} k^{-1/2}\right) + \mathcal{O}(t_0^{-1/4}) \\ &= 2 \sum_{k=1}^{\lfloor \sqrt{t/2\pi} \rfloor} \frac{\cos\left(\frac{t}{2} \log\left(\frac{t_0}{2\pi k^2}\right) - \frac{t_0}{2} - \frac{\pi}{8}\right)}{\sqrt{k}} + \mathcal{O}(t_0^{-1/4}) \end{aligned}$$

This expression differs from the expression of the proposition by a bounded number of terms, since

$$\left| \sqrt{\frac{t}{2\pi}} - \sqrt{\frac{t_0}{2\pi}} \right| = \frac{|t-t_0|}{\sqrt{2\pi t} + \sqrt{2\pi t_0}} = \mathcal{O}(t_0^{-1/4}) = \mathcal{O}(1),$$

and these terms are  $\mathcal{O}(t_0^{-1/4})$ . □

From the two last propositions, we deduce the following:

**Proposition 2.4.** *Let  $h > 0$ ,  $T \geq t_0 \geq 50(1 + h^4)$ . Then,*

$$\sup_{t \in [t_0 - h, t_0 + h]} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \ll \left( 1 + h \log T + \sum_{k \in \mathbb{Z}, |k| \leq h \log(T/2\pi)} \left| \zeta \left( \frac{1}{2} + i \left( t_0 + \frac{k\pi}{2 \log(T/2\pi)} \right) \right) \right|^2 \right. \\ \left. + \sum_{k \in \mathbb{Z}, |k| \leq T^{1/4}} \frac{\left| \zeta \left( \frac{1}{2} + i \left( t_0 + \frac{k\pi}{2 \log(T/2\pi)} \right) \right) \right|^2}{1 + |k|^3} \right).$$

*Proof.* We will prove the majorization with  $t_0^{1/4}$  instead of  $T^{1/4}$  in the last sum, which is stronger. With this change, and because of the inequality satisfied by  $t_0$ , all the values of  $t$  such that  $\zeta(\frac{1}{2} + it)$  is involved in the modified inequality are in the interval  $[t_0 - t_0^{1/4}, t_0 + t_0^{1/4}]$ . The previous proposition shows that for these values of  $t$ ,

$$Z(t) = H(t) + \mathcal{O}(t_0^{-1/4})$$

where  $H(t)$  is dominated by  $t_0^{1/4}$ , with a Fourier transform supported in  $[-\frac{1}{2} \log(t_0/2\pi), \frac{1}{2} \log(t_0/2\pi)]$ . We deduce

$$\left| \zeta \left( \frac{1}{2} + it \right) \right|^2 = (H(t))^2 + \mathcal{O}(1),$$

where the Fourier transform of  $(H(t))^2$  is supported in  $[-\log(t_0/2\pi), \log(t_0/2\pi)]$ , and a fortiori in  $[-\log(T/2\pi), \log(T/2\pi)]$ . Since we have a bounded error at each term when we replace  $|\zeta(\frac{1}{2} + it)|^2$  by  $(H(t))^2$ , it is sufficient to show an equality of the following form:

$$\sup_{t \in [t_0 - h, t_0 + h]} (H(t))^2 \ll \left( 1 + \sum_{k \in \mathbb{Z}, |k| \leq h \log(T/2\pi)} \left( H \left( t_0 + \frac{k\pi}{2 \log(T/2\pi)} \right) \right)^2 \right. \\ \left. + \sum_{k \in \mathbb{Z}, |k| \leq t_0^{1/4}} \frac{\left( H \left( t_0 + \frac{k\pi}{2 \log(T/2\pi)} \right) \right)^2}{1 + |k|^3} \right).$$

Now, this inequality is a consequence of Proposition 2.2, applied to  $f = H^2$ ,  $\lambda = \log(T/2\pi)$ ,  $A = 3$ ,  $R = t_0^{1/4}$ , since

$$\frac{\sup_{\mathbb{R}} H^2}{1 + (t_0^{1/4})^2} = \mathcal{O}(1).$$

□

We deduce the following bound on the maximum of  $|\zeta|$  on a random interval of fixed size, which, by applying Markov's inequality, completes the proof of Theorem 1.3:

**Proposition 2.5.** *Let  $U$  be a uniform variable on  $[0, 1]$ , and  $h > 0$ . Then, for all  $T \geq 10$ ,*

$$\mathbb{E} \left[ \sup_{t \in [UT - h, UT + h]} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \right] \ll_h (\log T)^2.$$

*Proof.* If we replace  $\ll$  by  $\ll_h$  in the previous proposition, we can assume only  $T \geq 10$  and  $T \geq t_0 \geq 0$ , instead of  $T \geq t_0 \geq 50(1 + h^4)$ . Indeed, the supremum in the left-hand side is bounded by a quantity depending only on  $h$  for  $0 \leq t_0 \leq 50(1 + h^4)$ . Hence, we can write:

$$\mathbb{E} \left[ \sup_{t \in [UT - h, UT + h]} \left| \zeta \left( \frac{1}{2} + iUT \right) \right|^2 \right]$$



$$\ll_h \left( 1 + h \log T + \sum_{k \in \mathbb{Z}, |k| \leq h \log(T/2\pi)} \mathbb{E} \left[ \left| \zeta \left( \frac{1}{2} + i \left( UT + \frac{k\pi}{2 \log(T/2\pi)} \right) \right) \right|^2 \right] \right. \\ \left. + \sum_{k \in \mathbb{Z}, |k| \leq T^{1/4}} \frac{\mathbb{E} \left[ \left| \zeta \left( \frac{1}{2} + i \left( UT + \frac{k\pi}{2 \log(T/2\pi)} \right) \right) \right|^2 \right]}{1 + |k|^3} \right).$$

Each expectation in the right-hand side is the average of  $|\zeta|^2$  on an interval of length  $T$  of the critical line, included in the interval

$$I(T, h) := \left[ \frac{1}{2} - \frac{i\pi}{2} \left( h + \frac{T^{1/4}}{\log(T/2\pi)} \right), \frac{1}{2} + iT + \frac{i\pi}{2} \left( h + \frac{T^{1/4}}{\log(T/2\pi)} \right) \right].$$

We deduce

$$\mathbb{E} \left[ \sup_{t \in [UT-h, UT+h]} \left| \zeta \left( \frac{1}{2} + iUT \right) \right|^2 \right] \\ \ll_h \left( 1 + h \log T + \frac{1 + h \log(T/2\pi) + \sum_{k \in \mathbb{Z}} \frac{1}{1+|k|^3}}{T} \int_{I(T, h)} |\zeta(s)|^2 |ds| \right).$$

By a classical result of Hardy and Littlewood on the second moment of  $\zeta$  (see [Tit86], Theorem 7.3), the last integral is dominated (with an implicit constant depending on  $h$ ) by  $T \log T$ , which gives the desired result.  $\square$

### 3 Averaging $\log \zeta$

It is known, from the Euler product and the series of the logarithm, that for  $\Re(s) > 1$ ,

$$\log \zeta(s) = \sum_{n \geq 1} \ell(n) n^{-s},$$

where  $\ell(n) = 1/k$  if  $n$  is a  $k$ -th power of a prime ( $k \geq 1$  integer), and  $\ell(n) = 0$  otherwise. If we apply this formula for  $s + it$  instead of  $s$ , and if we average by integrating with respect to  $\varphi(t)dt$ , then we get a sum in the right-hand side which involves the Fourier transform of  $\varphi$ . This is particularly interesting if  $\widehat{\varphi}$  is compactly supported, since the sum has finitely many non-zero terms in this case. More precisely, we have the following:

**Proposition 3.1.** *Let  $\varphi$  be an integrable function from  $\mathbb{R}$  to  $\mathbb{R}$ , such that  $\widehat{\varphi}$  is compactly supported. Then, for  $\Re(s) > 1$ , the following quantity:*

$$L_\varphi(s) = \int_{-\infty}^{\infty} \log \zeta(s + it) \varphi(t) dt$$

*is well-defined, and one has*

$$L_\varphi(s) = \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}(\log n),$$

*Moreover, the last formula defines an analytic continuation of  $L_\varphi$  to the whole complex plane.*

*Proof.* For  $\Re(s) > 1$  and  $t \in \mathbb{R}$ ,

$$\log \zeta(s + it) = - \sum_{p \in \mathcal{P}} \log(1 - p^{-s-it}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{p^{-k(s+it)}}{k} = \sum_{n \geq 1} \ell(n) n^{-s-it},$$

all the series being absolutely convergent and dominated by  $\sum_{n \geq 1} n^{-\Re(s)}$ , uniformly in  $t$  if  $s$  is fixed. Integrating in  $t$ , we get, using this domination and the fact that  $\varphi$  is integrable:

$$L_{\varphi}(s) = \sum_{n \geq 1} \ell(n) n^{-s} \int_{-\infty}^{\infty} n^{-it} \varphi(t) dt = \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}(\log n).$$

The last series has finitely many nonzero terms since  $\widehat{\varphi}$  is compactly supported, and then it defines an entire function extending  $L_{\varphi}$ .  $\square$

The result we have just proven gives some information on  $\log \zeta$  at points whose real part is strictly larger than 1. Of course, we are more interested in what happens on the critical line. To extend our previous result to the critical strip, we will use the principle of analytic continuation, but we need to be careful, since  $\log \zeta$  is not holomorphic everywhere because of the zeros and the pole of  $\zeta$ . However, if we assume the Riemann hypothesis, the only problem at the right of the critical line comes from the pole of  $\zeta$  at 1. The main result of the section is the following proposition, which shows how we can deal with this pole if the function  $\varphi$  satisfies some suitable extra assumptions:

**Proposition 3.2.** *Let us assume the Riemann hypothesis. Let  $\varphi$  be an even function from  $\mathbb{R}$  to  $\mathbb{R}$ , dominated by any negative power at infinity, whose integral is equal to 1, and whose Fourier transform is compactly supported. Then, for  $\sigma \in (1/2, 1)$ ,  $\tau \in \mathbb{R}$ ,  $H > 0$ ,*

$$\int_{-\infty}^{\infty} \log \zeta(\sigma + i(\tau + tH^{-1})) \varphi(t) dt = \sum_{n \geq 1} \ell(n) n^{-\sigma-i\tau} \widehat{\varphi}\left(\frac{\log n}{H}\right) + \mathcal{O}_{\varphi}\left(1 + \frac{e^{\mathcal{O}_{\varphi}(H)}}{1 + |\tau|}\right).$$

**Remark 3.3.** *Contrarily to the case  $\Re(s) > 1$ , the error term does not vanish in general, since one can check, from the discontinuity of the logarithm, that the left-hand side is not holomorphic in  $\sigma + i\tau$  if we allow  $\sigma$  to go below 1. Of course, the integral does not need that  $\log \zeta$  is well-defined at the left of the zeros or the pole.*

*Proof.* For  $\Re(s) > 1$ , we have, with the principal branch of the logarithm,

$$\log(s - 1) = \int_0^{\infty} \frac{e^{-\lambda} - e^{-\lambda(s-1)}}{\lambda} d\lambda.$$

Indeed, the integral is absolutely convergent, and the derivative of the integrand, with respect to  $s$ , is equal to  $e^{-\lambda(s-1)}$ , which is dominated by an integrable function on all compact subsets of  $\{s \in \mathbb{C}, \Re(s) > 1\}$ . Hence, the derivative in  $s$  of the integral is the integral of  $e^{-\lambda(s-1)}$ , which is  $1/(s-1)$ , giving the desired formula by comparing the values at  $s = 2$ . Using the previous proposition (with  $H\varphi(\cdot H)$  instead of  $\varphi$ ), we deduce, for  $\Re(s) > 1$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \log \zeta(s + itH^{-1}) \varphi(t) dt + \int_{-\infty}^{\infty} \log(s - 1 + itH^{-1}) \varphi(t) dt \\ &= \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}\left(\frac{\log n}{H}\right) + \int_{-\infty}^{\infty} \varphi(t) dt \int_0^{\infty} \frac{e^{-\lambda} - e^{-\lambda(s+itH^{-1}-1)}}{\lambda} d\lambda. \end{aligned}$$

The last double integral is absolutely convergent. Indeed, from the inequality

$$|e^u - 1| \leq \int_0^u |e^v| |dv| \leq |u| e^{(\Re(u))_+},$$

we get

$$\left| \frac{e^{-\lambda} - e^{-\lambda(s+itH^{-1}-1)}}{\lambda} \right| \leq 1 + |s + itH^{-1} - 1| e^{\lambda(\Re(1-s))_+},$$

and then

$$\int_{-\infty}^{\infty} |\varphi(t)| dt \int_0^1 \left| \frac{e^{-\lambda} - e^{-\lambda(s+itH^{-1}-1)}}{\lambda} \right| d\lambda < \infty$$

since  $\varphi$  is rapidly decaying at infinity by assumption: on the other hand,

$$\int_{-\infty}^{\infty} |\varphi(t)| dt \int_1^{\infty} \left| \frac{e^{-\lambda} - e^{-\lambda(s+itH^{-1}-1)}}{\lambda} \right| d\lambda < \infty$$

since  $|\varphi|$  is integrable, and the quantity in the second integral is, uniformly in  $t$ , exponentially decaying since  $\Re(s) > 1$ . By applying Fubini's theorem for the double integral, we deduce

$$\begin{aligned} & \int_{-\infty}^{\infty} \log[(s + itH^{-1} - 1)\zeta(s + itH^{-1})] \varphi(t) dt \\ &= \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi} \left( \frac{\log n}{H} \right) + \int_0^{\infty} \frac{e^{-\lambda} - \widehat{\varphi}(\lambda/H) e^{-\lambda(s-1)}}{\lambda} d\lambda. \end{aligned} \quad (3.1)$$

(recall that the integral of  $\varphi$  is 1). Here, we take the continuous version of  $z \mapsto \log[(z-1)\zeta(z)]$  on  $\Re > 1$  which is  $\log(z-1) + o(1)$  at  $+\infty$ . This continuous version, under the Riemann hypothesis, extends to an holomorphic function on  $\Re > 1/2$ . On the other hand, for  $\Re(s) \in (1/2, 2)$ , by using Theorem 9.2., Theorem 9.6. (A) of Titchmarsh and the Riemann hypothesis, we get, except in a small neighborhood of 1 (say  $\{s, |s-1| \leq 1/10\}$ ),

$$\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O} \left( \frac{\log(2 + |\Im(s)|)}{\Re(s) - (1/2)} \right).$$

From the Euler product,  $\zeta'(s)/\zeta(s)$  is uniformly bounded for  $\Re(s) \geq 2$ , and then for all  $s$  such that  $\Re(s) > 1/2$  and  $|s-1| > 1/10$ ,

$$\frac{d}{ds} \log[(s-1)\zeta(s)] = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} = \mathcal{O} \left( \frac{\log(2 + |\Im(s)|)}{1 \wedge (\Re(s) - (1/2))} \right).$$

By continuity of the left-hand side, the estimate remains true also for  $|s-1| \leq 1/10$ . Hence, for  $K$  compact of  $\Re > 1/2$ , we have, uniformly on  $s \in K$ ,

$$\frac{d}{ds} \log[(s + itH^{-1} - 1)\zeta(s + itH^{-1})] = \mathcal{O}_{K,H}(\log(2 + |t|)).$$

By differentiating under the integral, using the fact that  $\varphi$  is rapidly decaying, we deduce that

$$s \mapsto \int_{-\infty}^{\infty} \log[(s + itH^{-1} - 1)\zeta(s + itH^{-1})] \varphi(t) dt.$$

is holomorphic in  $\Re > 1/2$ . By the formula proven before, we have

$$\int_0^{\infty} \left| \frac{e^{-\lambda} - \widehat{\varphi}(\lambda/H) e^{-\lambda(s-1)}}{\lambda} \right| d\lambda < \infty$$

for  $\Re(s) > 1$ , and then for all  $s \in \mathbb{C}$ , by taking the difference of the integrand for different values of  $s$ , and by using the fact that  $|\widehat{\varphi}|$  is bounded and compactly supported. By differentiating under the integral (which is possible, again since  $\widehat{\varphi}$  is compactly supported), we deduce that

$$s \mapsto \int_0^{\infty} \frac{e^{-\lambda} - \widehat{\varphi}(\lambda/H) e^{-\lambda(s-1)}}{\lambda} d\lambda$$

is well-defined and holomorphic on the whole complex plane. By the principle of analytic continuation, we deduce that the equation (3.1) remains true for all  $s$  such that  $\Re(s) > 1/2$ . Now, for  $\Re(s) > 1/2$ ,  $s \notin (1/2, 1]$ , we get

$$\log \zeta(s) = \log[(s-1)\zeta(s)] - \log(s-1)$$

Indeed, the difference between the two quantities is a multiple of  $2i\pi$ , continuous in  $s$  for  $\Re(s) > 1/2$ ,  $s \notin (1/2, 1]$ , equal to 0 for  $\Re(s) > 1$ . Integrating (the integrals are absolutely convergent since  $\varphi$  is rapidly decreasing), and using (3.1), we deduce, for  $\Re(s) > 1$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \log[\zeta(s + itH^{-1})]\varphi(t)dt \\ &= \sum_{n \geq 1} \ell(n)n^{-s} \widehat{\varphi}\left(\frac{\log n}{H}\right) + \int_0^{\infty} \frac{e^{-\lambda} - \widehat{\varphi}(\lambda/H)e^{-\lambda(s-1)}}{\lambda} d\lambda \\ & - \int_{-\infty}^{\infty} \log(s + itH^{-1} - 1)\varphi(t)dt. \end{aligned}$$

We observe that for  $\Re(s) > 1$ , the two last terms cancel each other by the previous computation. It is not the case in general for  $\Re(s) \in (1/2, 1)$ . More precisely, since for  $z \notin \mathbb{R}$ ,

$$\log(-z) = (\log z) \pm i\pi = \log z + \mathcal{O}(1),$$

we deduce, for  $\Re(s) \in (1/2, 1)$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \log(s + itH^{-1} - 1)\varphi(t)dt &= \int_{-\infty}^{\infty} \log(1 - s - itH^{-1})\varphi(t)dt + \mathcal{O}(1) \\ &= \int_{-\infty}^{\infty} \varphi(t)dt \int_0^{\infty} \frac{e^{-\lambda} - e^{-\lambda(1-s-itH^{-1})}}{\lambda} d\lambda + \mathcal{O}(1). \\ &= \int_{-\infty}^{\infty} \frac{e^{-\lambda} - \widehat{\varphi}(-\lambda/H)e^{-\lambda(1-s)}}{\lambda} d\lambda + \mathcal{O}(1), \end{aligned}$$

this computation being justified similarly as for the case  $\Re(s) > 1$ . Since  $\widehat{\varphi}$  is even and compactly supported, we deduce

$$\begin{aligned} & \log[\zeta(s + itH^{-1})]\varphi(t)dt \\ &= \sum_{n \geq 1} \ell(n)n^{-s} \widehat{\varphi}\left(\frac{\log n}{H}\right) + \int_0^{AH} \frac{\widehat{\varphi}(\lambda/H)(e^{-\lambda(1-s)} - e^{-\lambda(s-1)})}{\lambda} d\lambda + \mathcal{O}(1). \end{aligned}$$

where  $A > 0$  is the largest point of the support of  $\widehat{\varphi}$ . In order to prove the result we have announced, it is then enough to show that

$$\int_0^A \widehat{\varphi}(u) \sinh(Hu(r + i\tau)) \frac{du}{u} = \mathcal{O}_{\varphi} \left( 1 + \frac{e^{\mathcal{O}_{\varphi}(H)}}{1 + |\tau|} \right),$$

for  $r \in (0, 1/2)$ . Now, let  $\Psi$  be the primitive of  $u \mapsto u^{-1} \sinh(Hu(r + i\tau))$  which vanishes at zero (this last function can be continuously extended at zero). Since  $\widehat{\varphi}$  is smooth ( $\varphi$  is rapidly decaying), and equal to zero at  $A$ , we get by integration by parts:

$$\int_0^A \widehat{\varphi}(u) \sinh(Hu(r + i\tau)) \frac{du}{u} = - \int_0^A \widehat{\varphi}'(u) \Psi(u) du \ll_{\varphi} \sup_{[0, A]} |\Psi|.$$

Now, for  $t \in [0, A]$ ,

$$\begin{aligned}\Psi(t) &= \int_{[0,t]} \sinh(Hu(r+i\tau)) \frac{du}{u} = \int_{[0,Ht(r+i\tau)]} \sinh(v) \frac{dv}{v} \\ &= \int_{[0,Ht\tau i]} \frac{\sinh v}{v} dv + \int_{[Ht\tau i, Ht\tau i + Htr]} \frac{\sinh v}{v} dv.\end{aligned}$$

The first integral is uniformly bounded, since the integral on  $\mathbb{R}_+$  of  $x \mapsto x^{-1} \sin x$  is semi-convergent. On the other hand,

$$\begin{aligned}\left| \frac{\sinh v}{v} \right| &\leq \frac{|e^v| + |e^{-v}|}{2|v|} \leq \frac{e^{|\Re(v)|}}{|v|}, \\ \left| \frac{\sinh v}{v} \right| &= \frac{1}{2} \left| \int_{-1}^1 e^{vx} dx \right| \leq e^{|\Re(v)|},\end{aligned}$$

and then

$$\left| \int_{[Ht\tau i, Ht\tau i + Htr]} \frac{\sinh v}{v} dv \right| \leq \int_{[Ht\tau i, Ht\tau i + Htr]} \frac{e^{|\Re(v)|}}{(1+|v|)/2} |dv| \ll \frac{e^{Htr}}{1+Ht|\tau|}(Htr)$$

The last quantity is bounded by

$$\frac{e^{Htr}}{Ht|\tau|}(Htr) = \frac{re^{Htr}}{|\tau|} \leq \frac{(1/2)e^{HA/2}}{|\tau|},$$

and by

$$Htre^{Htr} \leq e^{2Htr} \leq e^{HA}.$$

Hence, we have

$$\sup_{t \in [0, A]} |\Psi| \ll 1 + \frac{e^{HA}}{1+|\tau|},$$

which completes the proof of the proposition.  $\square$

The last proposition is only interesting if there exist functions  $\varphi$  satisfying the corresponding assumptions. Indeed:

**Proposition 3.4.** *There exists a function  $\varphi$ , real, nonnegative, even, dominated by any negative power at infinity, and such that its Fourier transform is compactly supported, takes values in  $[0, 1]$ , is even and equal to 1 at zero (which implies that the integral of  $\varphi$  is 1).*

*Proof.* Let  $\alpha$  be a nonnegative, smooth, compactly supported, even function whose  $L^2$  norm is equal to 1. We define  $\psi$  as the convolution of  $\alpha$  with itself:

$$\psi(x) = \int_{\mathbb{R}} \alpha(y)\alpha(x-y)dy.$$

It is clear that  $\psi$  is smooth, compactly supported, takes values in  $[0, 1]$  (by Cauchy-Schwarz inequality and the fact that  $\alpha$  is nonnegative), is even and equal to 1 at zero. We now define  $\varphi$  as the inverse Fourier transform of  $\psi$ . This function is real and even since it is the case for  $\psi$ , dominated by any power at infinity since  $\psi$  is smooth and compactly supported, and nonnegative since it is  $2\pi$  times the square of the inverse Fourier transform of  $\alpha$ , which is real since  $\alpha$  is real and even.  $\square$

## 4 The upper bound for the imaginary part

Similarly as what we have seen for  $\Re \log \zeta$ , the supremum of  $\Im \log \zeta$  on an interval can be controlled by its values at finitely many points. This comes from the fact that the argument of  $\zeta$  on the critical line has positive jumps of size  $\pi$  when we reach imaginary parts of zeros of  $\zeta$ , and decreases continuously, in a very well-controlled way, between the zeros of  $\zeta$ . We deduce that  $\Im \log \zeta$  cannot decrease too fast on the critical line:

**Proposition 4.1.** *For  $2 \leq t_1 \leq t_2$  which are not imaginary parts of zeros of  $\zeta$ , we have:*

$$\Im \log \zeta(1/2 + it_2) \geq \Im \log \zeta(1/2 + it_1) - (t_2 - t_1) \log t_2 + \mathcal{O}(1).$$

*Proof.* From Theorem 9.3. of [Tit86], we have for  $t \geq 2$ :

$$\Im \log \zeta(1/2 + it) = \pi N(t) - \frac{1}{2} t \log t + \frac{t(1 + \log(2\pi))}{2} + \mathcal{O}(1),$$

where  $N(t)$  denotes the number of zeros of  $\zeta$  with imaginary part in the interval  $(0, t]$ . Since  $N(t_1) \leq N(t_2)$ , we deduce

$$\begin{aligned} & \Im \log \zeta(1/2 + it_2) - \Im \log \zeta(1/2 + it_1) \\ & \geq -\frac{1}{2} [(t_2 \log t_2 - t_2) - (t_1 \log t_1 - t_1)] + (t_2 - t_1) \frac{\log(2\pi)}{2} + \mathcal{O}(1) \\ & \geq -\frac{1}{2} \int_{t_1}^{t_2} \log u \, du + \mathcal{O}(1), \end{aligned}$$

which proves the claim.  $\square$

From this result, we deduce that the argument of  $\zeta$  can be controlled by some of its averages, which will then imply, from Proposition 3.2, that it is also controlled by suitable finite sums indexed by primes.

**Proposition 4.2.** *Let  $\varphi$  be a function from  $\mathbb{R}$  to  $\mathbb{R}_+$  with integral 1, and decaying faster than any power at infinity. Let  $h > 0$ ,  $\epsilon \in (0, 1)$ . Then, for  $\tau > 3$  large enough depending only on  $h$ ,  $\epsilon$  and  $\varphi$ , and for  $(\log \tau)^{1/10} \leq H \leq \log \tau$ ,*

$$\begin{aligned} & \sup_{t \in [\tau - 2h, \tau + 2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) \, du \right| \\ & \geq (1 - \epsilon) \sup_{t \in [\tau - h, \tau + h]} |\Im \log \zeta(1/2 + it)| - \epsilon \sup_{t \in [\tau - 2h, \tau + 2h]} |\Im \log \zeta(1/2 + it)| - \mathcal{O}_{\varphi, \epsilon, h}(H^{-1} \log \tau). \end{aligned}$$

*Proof.* Let

$$\begin{aligned} M_1 &:= \sup_{t \in [\tau - h, \tau + h]} |\Im \log \zeta(1/2 + it)|, \\ M_2 &:= \sup_{t \in [\tau - 2h, \tau + 2h]} |\Im \log \zeta(1/2 + it)|. \end{aligned}$$

If  $M_1$  is the supremum of the positive part of  $\Im \log \zeta$ , then there exists  $t_0 \in [\tau - h, \tau + h]$  such that

$$\Im \log \zeta(1/2 + it_0) \geq M_1 - 1.$$

We can assume  $\tau > 2(1 + h)$ , which implies  $t_0 > 2$ , and then, for  $u > 0$ ,

$$\Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \geq M_1 - uH^{-1} \log(t_0 + uH^{-1}) + \mathcal{O}(1),$$

for  $u \in [-hH, 0]$ ,

$$\Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \geq -M_2,$$

and for  $u < -hH$ ,

$$|\Im \log \zeta(1/2 + i(t_0 + uH^{-1}))| \ll \log(2 + |t_0| + |u|H^{-1}),$$

the last estimate coming from Theorem 9.4. of [Tit86]. Let us now integrate these estimates with respect to  $\varphi(u - u_0)du$ , with  $u_0 > 0$  to be chosen later. The first estimate gives

$$\begin{aligned} & \int_0^\infty \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u - u_0) du \\ & \geq M_1 \int_{-u_0}^\infty \varphi(v) dv - H^{-1} \int_{-u_0}^\infty (v + u_0) [\log(1 + t_0) + \log(1 + |v|) + \log(1 + u_0)] \varphi(v) dv + \mathcal{O}(1), \end{aligned}$$

since  $H \geq (\log \tau)^{1/10} \geq (\log 3)^{1/10} > 1$ , then for  $v \geq -u_0$ ,

$$t_0 + (v + u_0)H^{-1} \leq t_0 + v + u_0 \leq (1 + t_0)(1 + |v|)(1 + u_0),$$

and the integral of  $\varphi$  is 1. Since  $\varphi$  is integrable with respect to  $(1 + |v|)(1 + \log(1 + |v|)) dv$ , we deduce

$$\begin{aligned} & \int_0^\infty \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u - u_0) du \\ & \geq M_1 \int_{-u_0}^\infty \varphi(v) dv - \mathcal{O}_{\varphi, u_0}(1 + H^{-1} \log t_0) \\ & \geq M_1 \int_{-u_0}^\infty \varphi(v) dv - \mathcal{O}_{\varphi, u_0}(H^{-1} \log \tau), \end{aligned}$$

the last line coming from the fact that on the one hand, we can assume  $\tau > 2h$  and then  $\log(t_0) = \log(\tau) + \mathcal{O}(1)$  for all  $t \in [\tau - h, \tau + h]$ , and on the other hand,  $H^{-1} \log \tau \geq 1$  by assumption on  $H$ . The second estimate of  $\Im \log \zeta$  gives

$$\int_{-hH}^0 \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u - u_0) du \geq -M_2 \int_{-\infty}^{-u_0} \varphi(v) dv.$$

Finally, the last estimate gives

$$\begin{aligned} & \int_{-\infty}^{-hH} \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u - u_0) du \\ & \geq -\mathcal{O} \left( \int_{-\infty}^{-hH - u_0} [\log(2 + t_0) + \log(2 + u_0) + \log(2 + |v|)] \varphi(v) dv \right) \\ & \geq -\mathcal{O}_{\varphi, u_0, h, A}(H^{-A} \log t_0) = -\mathcal{O}_{\varphi, u_0, h, A}(H^{-A} \log \tau), \end{aligned}$$

for any  $A > 0$ , since  $\varphi$  is rapidly decaying at  $-\infty$  by assumption. Since  $H \geq (\log \tau)^{1/10}$ , we obtain, by taking  $A = 10$ , a lower bound  $-\mathcal{O}_{\varphi, u_0, h}(1)$ . Adding the three integrals on the intervals  $(-\infty, -hH]$ ,  $[-hH, 0]$  and  $[0, \infty)$ , and translating the interval of integration, we deduce

$$\begin{aligned} & \int_{-\infty}^\infty \Im \log \zeta(1/2 + i(t_0 + u_0H^{-1} + uH^{-1})) \varphi(u) du \\ & \geq M_1 \int_{-u_0}^\infty \varphi(v) dv - M_2 \int_{-\infty}^{-u_0} \varphi(v) dv - \mathcal{O}_{\varphi, u_0, h}(H^{-1} \log \tau). \end{aligned}$$

We can now choose  $u_0$  depending only on  $\varphi$  and  $\epsilon$ , sufficiently large in order to have

$$\int_{-\infty}^{-u_0} \varphi(v) dv \leq \epsilon.$$

Then, by taking  $t_1 = t_0 + u_0 H^{-1} = t_0 + \mathcal{O}_{\varphi, \epsilon}(H^{-1})$ , we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t_1 + uH^{-1})) \varphi(u) du \\ & \geq M_1(1 - \epsilon) - M_2\epsilon - \mathcal{O}_{\varphi, \epsilon, h}(H^{-1} \log \tau). \end{aligned}$$

If  $\tau$  is large enough depending on  $h$ ,  $\epsilon$  and  $\varphi$ , then  $H \geq (\log \tau)^{1/10}$  can be assumed to be sufficiently large in order to have

$$t_1 - t_0 = \mathcal{O}_{\varphi, \epsilon}(H^{-1}) \leq h,$$

and then  $t_1 \in [\tau - 2h, \tau + 2h]$ .

This proves the proposition in the case where  $M_1$  is the supremum of the positive part of  $\Im \log \zeta$ .

If  $M_1$  is the supremum of the negative part of  $\Im \log \zeta$ , then there exists  $t_0 \in [\tau - h, \tau + h]$  such that

$$\Im \log \zeta(1/2 + it_0) \leq -M_1 + 1.$$

We can assume  $\tau > 2(1 + h)$ , which implies for  $u \in [-hH, 0]$ ,  $t_0 + uH^{-1} \geq (\tau - h) - h > 2$ , and then,

$$\Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \leq -M_1 + |u|H^{-1} \log t_0 + \mathcal{O}(1),$$

for  $u \in [0, hH]$ ,

$$\Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \leq M_2,$$

and for  $|u| > -hH$ ,

$$|\Im \log \zeta(1/2 + i(t_0 + uH^{-1}))| \ll \log(2 + |t_0| + |u|H^{-1}),$$

We now integrate these estimates with respect to  $\varphi(u + u_1)du$ , with  $u_1 > 0$ . The first estimate gives

$$\begin{aligned} & \int_{-hH}^0 \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u + u_1) du \\ & \leq -M_1 \int_{u_1 - hH}^{u_1} \varphi(v) dv + H^{-1} \int_{u_1 - hH}^{u_1} (|v| + u_1)(\log t_0) \varphi(v) dv + \mathcal{O}(1), \\ & \leq -M_1 \int_{u_1 - hH}^{u_1} \varphi(v) dv + \mathcal{O}_{\varphi, u_1}(1 + H^{-1} \log t_0) \\ & \leq -M_1 \int_{u_1 - hH}^{u_1} \varphi(v) dv + \mathcal{O}_{\varphi, u_1}(H^{-1} \log \tau). \end{aligned}$$

The second estimate of  $\Im \log \zeta$  gives

$$\int_0^{hH} \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u + u_1) du \leq M_2 \int_{u_1}^{u_1 + hH} \varphi(v) dv.$$

Finally, the last estimates gives

$$\int_{\mathbb{R} \setminus [-hH, hH]} \Im \log \zeta(1/2 + i(t_0 + uH^{-1})) \varphi(u + u_1) du$$



$$\begin{aligned}
&\leq \mathcal{O} \left( \int_{\mathbb{R} \setminus [-hH+u_1, hH+u_1]} [\log(2+t_0) + \log(2+u_1) + \log(2+|v|)] \varphi(v) dv \right) \\
&\leq \mathcal{O}_{\varphi, u_1, h} (H^{-10} \log t_0) = \mathcal{O}_{\varphi, u_1, h} (H^{-10} \log \tau) = \mathcal{O}_{\varphi, u_0, h} (1),
\end{aligned}$$

provided that  $hH - u_1 \geq hH/2$ , i.e.  $hH \geq 2u_1$ . We then get, under this last condition:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t_0 - u_1 H^{-1} + u H^{-1})) \varphi(u) du \\
&\leq -M_1 \int_{u_1 - hH}^{u_1} \varphi(v) dv + M_2 \int_{u_1}^{u_1 + hH} \varphi(v) dv + \mathcal{O}_{\varphi, u_1, h}(H^{-1} \log \tau) \\
&\leq -M_1 \int_{-u_1}^{u_1} \varphi(v) dv + M_2 \int_{u_1}^{\infty} \varphi(v) dv + \mathcal{O}_{\varphi, u_1, h}(H^{-1} \log \tau).
\end{aligned}$$

We can now choose  $u_1$  depending only on  $\varphi$  and  $\epsilon$ , such that

$$\int_{-u_1}^{u_1} \varphi(v) dv \geq 1 - \epsilon.$$

Then, by taking  $t_1 = t_0 - u_1 H^{-1} = t_0 - \mathcal{O}_{\varphi, \epsilon}(H^{-1})$ , we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t_1 + u H^{-1})) \varphi(u) du \\
&\leq -M_1(1 - \epsilon) + M_2 \epsilon + \mathcal{O}_{\varphi, \epsilon, h}(H^{-1} \log \tau).
\end{aligned}$$

If  $\tau$  is large enough depending on  $h$ ,  $\epsilon$  and  $\varphi$ , then  $H \geq (\log \tau)^{1/10}$  can be assumed to be sufficiently large in order to have  $hH \geq 2u_1$  and  $t_1 \in [\tau - 2h, \tau + 2h]$ . This completes the proof of the proposition.  $\square$

We deduce the following probabilistic result:

**Proposition 4.3.** *Let  $\varphi$  be as in the previous proposition, let  $\epsilon \in (0, 1)$ ,  $A \geq 1$ ,  $h \geq 0$ . For  $T > 100$ , let  $H := (\log T)(\log \log \log T)^{1/2}(\log \log T)^{-1}$ . Then, for  $U$  random, uniform on  $[0, 1]$ , we have*

$$\begin{aligned}
&\mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A(1 + \epsilon) \log \log T \right] \\
&\leq \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + u H^{-1})) \varphi(u) du \right| \geq A \log \log T \right] \\
&+ 2 \mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq 10A(1 + \epsilon) \log \log T \right] + \mathcal{O}_{\varphi, h, \epsilon}(T^{-0.99}).
\end{aligned}$$

**Remark 4.4.** *The implied constant does not depend on  $A$ .*

*Proof.* Except on an event of probability  $\mathcal{O}_{\varphi, h, \epsilon}(T^{-1})$ ,  $\tau = UT$  is large enough in order to apply the previous proposition. Moreover, by changing the implicit constant in  $\mathcal{O}_{\varphi, h, \epsilon}(T^{-0.99})$ , we can assume that  $T$  is sufficiently large, in order to have  $(\log \tau)^{1/10} \leq (\log T)^{1/10} \leq H$ . We also have, for  $T$  large enough:

$$\begin{aligned}
\mathbb{P}[H > \log \tau] &= \mathbb{P} \left[ \log(UT) < (\log T)(\log \log \log T)^{1/2}(\log \log T)^{-1} \right] \\
&\leq \mathbb{P} \left[ \log(UT) < \frac{1}{100} \log T \right] = \mathbb{P}[UT < T^{1/100}] = T^{-0.99}.
\end{aligned}$$

Applying the previous proposition (with a different value of  $\epsilon$ ), we deduce that, outside an event of probability  $\mathcal{O}_{\varphi,h,\epsilon}(T^{-0.99})$ ,

$$\begin{aligned} & \sup_{t \in [\tau-2h, \tau+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \\ & \geq (1 - \epsilon) \sup_{t \in [\tau-h, \tau+h]} |\Im \log \zeta(1/2 + it)| - \epsilon \sup_{t \in [\tau-2h, \tau+2h]} |\Im \log \zeta(1/2 + it)| - \mathcal{O}_{\varphi,h,\epsilon}(H^{-1} \log T), \end{aligned}$$

for  $\tau = UT$ . If  $\epsilon < 1/12$ , if the second supremum is at least  $A(\log \log T)/(1 - 12\epsilon)$ , if the last supremum is at most  $10A(\log \log T)/(1 - 12\epsilon)$ , and if we exclude an event of probability  $\mathcal{O}_{\varphi,h,\epsilon}(T^{-0.99})$ , we get

$$\begin{aligned} & \sup_{t \in [\tau-2h, \tau+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \\ & \geq \frac{A(\log \log T)(1 - 11\epsilon)}{1 - 12\epsilon} - \mathcal{O}_{\varphi,h,\epsilon}((\log \log \log T)^{-1/2} \log \log T) \\ & \geq A \log \log T, \end{aligned}$$

if  $T$  is large enough depending on  $\varphi, \epsilon, h$  (not on  $A$  since we assume  $A \geq 1$ ), which can be assumed by changing the implicit constant in  $\mathcal{O}_{\varphi,h,\epsilon}(T^{-0.99})$ . We then get:

$$\begin{aligned} & \mathbb{P} \left[ \sup_{t \in [\tau-h, \tau+h]} |\Im \log \zeta(1/2 + it)| \geq \frac{A \log \log T}{1 - 12\epsilon} \right] \\ & \leq \mathbb{P} \left[ \sup_{t \in [\tau-2h, \tau+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq A \log \log T \right] \\ & + \mathbb{P} \left[ \sup_{t \in [\tau-2h, \tau+2h]} |\Im \log \zeta(1/2 + it)| \geq \frac{10A \log \log T}{1 - 12\epsilon} \right] + \mathcal{O}_{\varphi,h,\epsilon}(T^{-0.99}). \\ & \leq \mathbb{P} \left[ \sup_{t \in [\tau-2h, \tau+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq A \log \log T \right] \\ & + \mathbb{P} \left[ \sup_{t \in [\tau'-h, \tau'+h]} |\Im \log \zeta(1/2 + it)| \geq \frac{10A \log \log T}{1 - 12\epsilon} \right] \\ & + \mathbb{P} \left[ \sup_{t \in [\tau''-h, \tau''+h]} |\Im \log \zeta(1/2 + it)| \geq \frac{10A \log \log T}{1 - 12\epsilon} \right] + \mathcal{O}_{\varphi,h,\epsilon}(T^{-0.99}), \end{aligned}$$

where  $\tau' = \tau + h$  and  $\tau'' = \tau - h$ . Since  $\tau$  is uniform on  $[0, T]$ , it is possible to define random variables  $\tau_1$  and  $\tau_2$  with the same laws as  $\tau'$  and  $\tau''$ , and both equal to  $\tau$  with probability at least  $1 - (h/T)$  (for example, take  $\tau_1 = \tau + T\mathbb{1}_{\tau < h}$  and  $\tau_2 = \tau - T\mathbb{1}_{\tau > T-h}$ ). We deduce that we can replace  $\tau'$  and  $\tau''$  by  $\tau_1$  and  $\tau_2$ , and then both by  $\tau$ , in the previous estimate. We then get the claim in the proposition, with  $\epsilon$  replaced by  $(12\epsilon)/(1 - 12\epsilon)$ . Since this quantity can take any value in  $(0, 1)$ , we are done.  $\square$

We then deduce the following:

**Proposition 4.5.** *With the notation of the previous proposition:*

$$\mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A(1 + \epsilon) \log \log T \right]$$

$$\begin{aligned}
&\leq \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq A \log \log T \right] \\
&+ (\log T) \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq 10 \log \log T \right] \\
&+ \mathcal{O}_{\varphi, h, \epsilon}(T^{-0.98}).
\end{aligned}$$

*Proof.* We can iterate the result of the previous proposition, which gives, for  $k_0 \geq 1$  integer:

$$\begin{aligned}
&\mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A(1 + \epsilon) \log \log T \right] \\
&\leq \sum_{k=0}^{k_0-1} 2^k \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq 10^k A \log \log T \right] \\
&+ 2^{k_0} \mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq 10^{k_0} A(1 + \epsilon) \log \log T \right] + \mathcal{O}_{\varphi, h, \epsilon}(2^{k_0} T^{-0.99}).
\end{aligned}$$

We now take  $k_0 = 1 + \lfloor \log \log T \rfloor$ , which implies that the last probability is zero for  $T$  large enough depending on  $h$ , since

$$\sup_{t \in [-h, T+h]} |\Im \log \zeta(1/2 + it)| \ll_h \log T,$$

whereas

$$10^{k_0} A(1 + \epsilon) \log \log T \geq 10^{k_0} \geq (\log T)^{\log 10}.$$

For  $T$  large enough,  $2^{k_0} T^{-0.99} \leq T^{-0.98}$ . In the sum in  $k$ , each term corresponding to  $k > 0$  is at most

$$2^k \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq 10 \log \log T \right],$$

since  $A \geq 1$ . Hence, the sum of the terms for  $k = 1$  to  $k_0 - 1$  is at most

$$(2^{k_0} - 2) \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq 10 \log \log T \right],$$

where

$$2^{k_0} - 2 \leq 2^{1+\log \log T} = 2(\log T)^{\log 2} \leq \log T$$

for  $T$  large enough. Adding the term corresponding to  $k = 0$  gives the desired result.  $\square$

The properties of the imaginary part of  $\zeta$  on the critical line imply that the previous proposition can be rewritten as follows:

**Proposition 4.6.** *Let  $\varphi$  be as in the previous proposition, let  $A > B \geq 1$ ,  $h > 0$ . For  $T > 100$ , let  $H := (\log T)(\log \log \log T)^{1/2}(\log \log T)^{-1}$ . Then, for  $U$  random, uniform on  $[0, 1]$ ,*

$$\begin{aligned}
&\mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A \log \log T \right] \\
&\leq (\log T) \left( \sup_{d \in [-2h, 2h]} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq B \log \log T \right] \right)
\end{aligned}$$

$$\begin{aligned}
& + (\log T)^2 \left( \sup_{d \in [-2h, 2h]} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq 9 \log \log T \right] \right) \\
& + \mathcal{O}_{\varphi, h, A, B}(T^{-0.98}).
\end{aligned}$$

*Proof.* Let  $A' := (A + B)/2$ . Applying the previous proposition to  $A'$  instead of  $A$ , and  $\epsilon \leq (A/A') - 1$ , depending only on  $A$  and  $B$ , we get

$$\begin{aligned}
& \mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A \log \log T \right] \\
& \leq \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq A' \log \log T \right] \\
& + (\log T) \mathbb{P} \left[ \sup_{t \in [UT-2h, UT+2h]} \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t + uH^{-1})) \varphi(u) du \right| \geq 10 \log \log T \right] \\
& + \mathcal{O}_{\varphi, h, A, B}(T^{-0.98}).
\end{aligned}$$

Discarding an event of probability  $\mathcal{O}_h(T^{-1})$ , which can be absorbed in the error term, we can assume that  $[UT-3h, UT+3h]$  is included in  $[2, T]$ . Hence, for  $UT-2h \leq t_1 < t_2 \leq UT+2h$  and  $u \in [-hH, hH]$ , we get from Proposition 4.1,

$$\Im \log \zeta(1/2 + i(t_2 + uH^{-1})) \geq \Im \log \zeta(1/2 + i(t_1 + uH^{-1})) - (t_2 - t_1) \log T + \mathcal{O}(1).$$

If  $T$  is large enough (which implies  $H \geq 1$ ), we also get, for  $u \notin [-hH, hH]$ ,

$$\Im \log \zeta(1/2 + i(t_2 + uH^{-1})) \geq \Im \log \zeta(1/2 + i(t_1 + uH^{-1})) - \mathcal{O}(\log(T + |u|))$$

by the classical bound of  $\Im \log \zeta$  on the critical line (Theorem 9.4. of [Tit86]). Integrating with respect to  $\varphi(u)du$ , using the rapid decay of  $\varphi$  and the fact that  $H \geq (\log T)^{1/10}$ , we deduce, for  $T$  large enough,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t_2 + uH^{-1})) \varphi(u) du & \geq \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(t_1 + uH^{-1})) \varphi(u) du \\
& - (t_2 - t_1) \log T + \mathcal{O}_h(1).
\end{aligned}$$

Hence, for  $T$  large enough, the extrema of the last integral for  $t_1 \in [UT-2h, UT+2h]$  are controlled, up to an error  $\mathcal{O}_h(1)$ , by the highest and the lowest values corresponding to  $t_1 = UT-2h + 4hk/(\lfloor \log T \rfloor - 1)$ , where  $k \in \{0, 1, \dots, \lfloor \log T \rfloor - 1\}$ . A union bound then gives:

$$\begin{aligned}
& \mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A \log \log T \right] \\
& \leq \sum_{k=0}^{\lfloor \log T \rfloor - 1} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT - 2h + 4hk/(\lfloor \log T \rfloor - 1) + uH^{-1})) \varphi(u) du \right| \right. \\
& \quad \left. \geq A' \log \log T + \mathcal{O}_h(1) \right] \\
& + (\log T) \sum_{k=0}^{\lfloor \log T \rfloor - 1} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT - 2h + 4hk/(\lfloor \log T \rfloor - 1) + uH^{-1})) \varphi(u) du \right| \right. \\
& \quad \left. \geq 10 \log \log T + \mathcal{O}_h(1) \right] \\
& + \mathcal{O}_{\varphi, h, A, B}(T^{-0.98}).
\end{aligned}$$

By changing the implicit constant in  $\mathcal{O}_{\varphi,h,A,B}(T^{-0.98})$ , we can assume  $T$  large enough, depending on  $h$ , in order to have

$$\begin{aligned} A' \log \log T + \mathcal{O}_h(1) &\geq B \log \log T, \\ 10 \log \log T + \mathcal{O}_h(1) &\geq 9 \log \log T, \end{aligned}$$

which then implies the result of the proposition.  $\square$

It remains to estimate the probability involved in the last proposition. First, it is possible to split the probability into three pieces, involving sums on primes we discussed previously:

**Proposition 4.7.** *Let  $B > 1$ ,  $B_1, B_2, B_3 > 0$  such that  $B_1 + B_2 + B_3 < B$ . Let us take the notation of the previous proposition, and let us assume that  $\varphi$  is even, with compactly supported Fourier transform. Then, under the Riemann hypothesis, and for any  $R > 0$ ,  $d \in [-2h, 2h]$ ,*

$$\begin{aligned} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq B \log \log T \right] \\ \leq P_1 + P_2 + P_3 + \mathcal{O}_{\varphi,h,B,B_1,B_2,B_3}(T^{-0.99}), \end{aligned}$$

where

$$\begin{aligned} P_1 &= \mathbb{P} \left[ \left| \sum_{p \in \mathcal{P}, p \leq R} p^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log p}{H} \right) \right| \geq B_1 \log \log T \right], \\ P_2 &= \mathbb{P} \left[ \left| \sum_{p \in \mathcal{P}, p > R} p^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log p}{H} \right) \right| \geq B_2 \log \log T \right], \\ P_3 &= \mathbb{P} \left[ \frac{1}{2} \left| \sum_{p \in \mathcal{P}} p^{-1-2i(UT+d)} \widehat{\varphi} \left( \frac{2 \log p}{H} \right) \right| \geq B_3 \log \log T \right]. \end{aligned}$$

*Proof.* By using Proposition 3.2, and by passing to the limit  $\sigma \rightarrow 1/2$  (which is possible with the imaginary part, by using dominated convergence and the fact, deduced from Theorem 9.6. (B) of [Tit86], that  $\Im \log \zeta(\sigma + it) \ll \log(2 + |t|)$  for  $1/2 \leq \sigma \leq 3/4$  and  $t \in \mathbb{R}$ ), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \\ = \Im \sum_{n \geq 1} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) + \mathcal{O}_{\varphi} \left( 1 + \frac{e^{\mathcal{O}_{\varphi}(H)}}{1 + |UT + d|} \right). \end{aligned}$$

Discarding and event of probability  $\mathcal{O}_h(T^{-0.99})$ , we can assume that  $UT + d \geq T^{1/100}$ , which implies, since  $H = (\log T)(\log \log T)^{1/2}(\log \log T)^{-1}$ , that the error term is  $\mathcal{O}_{\varphi}(1)$  for  $T$  large enough depending on  $\varphi$ , and then smaller than  $B' \log \log T$  for  $T$  large enough depending on  $\varphi, B, B_1, B_2, B_3$ , where

$$B' := \frac{B - (B_1 + B_2 + B_3)}{2} > 0.$$

The last condition on  $T$  can be assumed by changing the implicit constant in  $\mathcal{O}_{\varphi,h,B,B_1,B_2,B_3}(T^{-0.99})$ , and then it is enough to show:

$$\mathbb{P} \left[ \left| \Im \sum_{n \geq 1} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq (B - B') \log \log T \right]$$

$$\leq P_1 + P_2 + P_3 + \mathcal{O}_{\varphi, h, B, B_1, B_2, B_3}(T^{-0.99}).$$

Now, we have

$$\begin{aligned} & \mathbb{P} \left[ \left| \Im \sum_{n \geq 1} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq (B - B') \log \log T \right] \\ & \leq \mathbb{P} \left[ \left| \sum_{n \geq 1} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq (B - B') \log \log T \right] \\ & \leq \mathbb{P} \left[ \left| \sum_{n \geq 1, \ell(n)=1, n \leq R} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq B_1 \log \log T \right] \\ & + \mathbb{P} \left[ \left| \sum_{n \geq 1, \ell(n)=1, n > R} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq B_2 \log \log T \right] \\ & + \mathbb{P} \left[ \left| \sum_{n \geq 1, \ell(n)=1/2} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq B_3 \log \log T \right] \\ & + \mathbb{P} \left[ \left| \sum_{n \geq 1, 0 < \ell(n) < 1/2} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \geq B_4 \log \log T \right], \end{aligned}$$

for

$$B_4 := B - B' - B_1 - B_2 - B_3 = \frac{B - (B_1 + B_2 + B_3)}{2} > 0.$$

In this sum of four probabilities, the first one is  $P_1$ , the second is  $P_2$ , the third is  $P_3$  and the fourth is equal to 0 for  $T$  large enough depending only on  $B_4$ , since

$$\left| \sum_{n \geq 1, 0 < \ell(n) < 1/2} \ell(n) n^{-1/2-i(UT+d)} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right| \leq \sum_{r \geq 3} \sum_{p \in \mathcal{P}} p^{-r/2}$$

is uniformly bounded (recall that  $|\widehat{\varphi}| \leq 1$  since  $\varphi$  is nonnegative with integral 1).  $\square$

We will now estimate the probabilities  $P_1, P_2, P_3$ .

**Proposition 4.8.** *With the notation above, for  $R := e^{\log T / (2 \log \log T)}$  and  $B_1 > 1$ ,*

$$P_1 \ll_{B_1} (\log T)^{-1 - \log B_1}.$$

*Proof.* We apply a result by Soundararajan ([Sou09], Lemma 3), by taking, with the notation of this paper,  $a(p) = p^{i(T-d)} \widehat{\varphi}(H^{-1} \log p)$ . By Markov's inequality, we deduce, for  $T$  large enough, and  $k \geq 0$  integer such that  $R^k \leq T / \log T$ ,

$$P_1 \ll (B_1 \log \log T)^{-2k} k! \left( \sum_{p \in \mathcal{P}, p \leq R} p^{-1} \right)^k.$$

For  $T$  large enough, it is possible to take  $k = \lfloor \log \log T \rfloor$ , and then

$$\begin{aligned} P_1 & \ll (B_1 \log \log T)^{-2 \log \log T + 2} (e^{-1} \log \log T)^{\log \log T} \sqrt{\log \log T} (\log \log T + \mathcal{O}(1))^{\log \log T} \\ & = B_1^{-2 \log \log T + 2} (\log \log T)^{5/2} e^{-\log \log T} (1 + \mathcal{O}((\log \log T)^{-1}))^{\log \log T} \end{aligned}$$

$$\begin{aligned} &\ll B_1^2 (\log T)^{-2 \log B_1 - 1} (\log \log T)^{5/2} \\ &\ll_{B_1} (\log T)^{-\log B_1 - 1}. \end{aligned}$$

By changing the implicit constant, we can remove the assumption that  $T$  is large.  $\square$

**Proposition 4.9.** *With the notation above, for  $R := e^{\log T / (2 \log \log T)}$  and  $B_2 > 0$ ,*

$$P_2 \ll_{\varphi, B_2} (\log T)^{-10}.$$

*Proof.* We apply the same lemma by Soundararajan, now with  $a(p) = p^{i(T-d)} \widehat{\varphi}(H^{-1} \log p) \mathbf{1}_{p > R}$ . We observe that  $a(p) = 0$  for all  $p \geq e^{\mathcal{O}_\varphi(H)}$ , since  $\widehat{\varphi}$  is compactly supported. For  $T$  large enough depending on  $\varphi$ , we can then apply the lemma for  $k = \lfloor 100 \log \log T (\log \log \log T)^{-1} \rfloor$ , since under this assumption,

$$(e^{\mathcal{O}_\varphi(H)})^k \leq e^{\mathcal{O}_\varphi((\log \log \log T)^{-1/2} \log T)} \leq \frac{T}{\log T}.$$

For  $T$  large enough depending on  $\varphi$ , we have  $k \leq \log \log T$ , and

$$\sum_{p \in \mathcal{P}} \frac{|a(p)|^2}{p} \leq \sum_{R < p \leq e^{\mathcal{O}_\varphi(H)}} p^{-1} \leq \sum_{e^{\log T / (2 \log \log T)} < p \leq T} p^{-1} \ll \log \log \log T.$$

Hence,

$$\begin{aligned} P_2 &\ll (B_2 \log \log T)^{-2k} k^k (\mathcal{O}(\log \log \log T))^k \\ &\leq (\mathcal{O}_{B_2}(\log \log \log T))^k (\log \log T)^{-k} \\ &= \exp \left( (100 \log \log T (\log \log \log T)^{-1} + \mathcal{O}(1)) (-\log \log \log T + \log \log \log \log T + \mathcal{O}_{B_2}(1)) \right) \\ &\leq (\log T)^{-10}, \end{aligned}$$

for  $T$  large enough depending on  $\varphi$  and  $B_2$ . This proves the proposition.  $\square$

**Proposition 4.10.** *With the notation above, for  $B_3 > 0$ ,*

$$P_3 \ll_{\varphi, B_3} (\log T)^{-10}.$$

*Proof.* We use the same lemma as before, for  $2T$  instead of  $T$ , and  $a(p) = \frac{1}{2} p^{-1/2+2i(T-d)} \widehat{\varphi}(2H^{-1} \log p)$ . Again,  $a(p) = 0$  for  $p \geq e^{\mathcal{O}_\varphi(H)}$ , and then we can again take  $k = \lfloor 100 \log \log T (\log \log \log T)^{-1} \rfloor$  for  $T$  large enough depending on  $\varphi$ . Moreover,

$$\sum_{p \in \mathcal{P}} \frac{|a(p)|^2}{p} \leq \sum_{p \in \mathcal{P}} p^{-2} \ll 1 \ll \log \log \log T,$$

which implies that an exactly similar computation as in the previous proof gives the result of the proposition.  $\square$

We deduce the leading order of an upper bound for  $\Im \log \zeta$ :

**Proposition 4.11.** *Let us assume the Riemann hypothesis. Then, for all  $A > 1$ ,  $h > 0$ ,  $U$  random, uniform on  $[0, 1]$ ,*

$$\mathbb{P} \left[ \sup_{t \in [UT-h, UT+h]} |\Im \log \zeta(1/2 + it)| \geq A \log \log T \right] \xrightarrow{T \rightarrow \infty} 0.$$

*Proof.* We choose arbitrarily a function  $\varphi$  satisfying all the assumptions given in this section, which is possible by Proposition 3.4. By Propositions 4.7, 4.8, 4.9, 4.10, we have for all  $B > 1$ ,  $B_1 > 1$ ,  $B_2, B_3 > 0$  such that  $B_1 + B_2 + B_3 < B$ ,

$$\sup_{d \in [-2h, 2h]} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq B \log \log T \right] \\ \ll_{\varphi, h, B, B_1, B_2, B_3} (\log T)^{-1 - \log B_1} + (\log T)^{-10} + T^{-0.99}$$

By taking  $B_1 = \sqrt{B}$ ,  $B_2 = B_3 = (B - \sqrt{B})/3$ , we deduce, for  $1 < B \leq 100$ ,

$$\sup_{d \in [-2h, 2h]} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq B \log \log T \right] \\ \ll_{\varphi, h, B} (\log T)^{-1 - (1/2) \log B},$$

and in particular,

$$\sup_{d \in [-2h, 2h]} \mathbb{P} \left[ \left| \int_{-\infty}^{\infty} \Im \log \zeta(1/2 + i(UT + d + uH^{-1})) \varphi(u) du \right| \geq 9 \log \log T \right] \\ \ll_{\varphi, h, B} (\log T)^{-1 - (1/2) \log 9} \ll (\log T)^{-2.09}.$$

We then conclude the proof of the present proposition by taking (say)  $B = \min(\sqrt{A}, 100)$  and by applying Proposition 4.6.  $\square$

## 5 A lower bound of the supremum in term of sums on primes

In this section, the quantity we will bound from below is the supremum of the positive part of  $\Re(\kappa \log \zeta(1/2 + i\tau))$ , for  $\tau \in [UT - h, UT + h]$ ,  $\kappa \in \{1, i, -i\}$ ,  $U$  uniform on  $[0, 1]$ . The parameter  $\kappa$  is used to deal with the real part and the imaginary part of  $\log \zeta$  at the same time. The facts we use are the following: on the one hand, averaging  $\log \zeta$  makes this quantity smoother and then tends to decrease its supremum, on the other hand, by Proposition 3.2, it gives something related to finite sums over primes. The goal of this section is to prove the following proposition:

**Proposition 5.1.** *Let us assume the Riemann hypothesis. Then, for  $T > 10$ ,  $h > 0$ ,  $U$  uniform on  $[0, 1]$ ,  $H$  integer such that  $(\log(3 + T))^{1/10} \leq H \leq \frac{\log T}{(\log \log T)^2}$ , we have, with probability tending to 1 when  $T$  goes to infinity:*

$$\sup_{\tau \in [UT - h, UT + h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \\ \geq \sup_{k \in \{0, 1, \dots, H-1\}} \Re \left( \kappa \sum_{p \in \mathcal{P} \cap [1, e^H]} p^{-\frac{1}{2} - i(TU - h/2 + kh/H)} \right) + \mathcal{O}_h(\sqrt{\log \log T}).$$

The first step of the proof consists in bounding the left-hand side from below by a series which is similar to the right-hand side, but with smooth cutoff instead of sharp cutoff at  $e^H$ . Such smooth cutoff is naturally obtained by using Proposition 3.2.

**Proposition 5.2.** *Let us assume the Riemann hypothesis. Let  $\varphi$  be a real, nonnegative, even function, dominated by any negative power at infinity, and such that its Fourier transform*



$\psi := \widehat{\varphi}$  is compactly supported, takes values in  $[0, 1]$ , is even and equal to 1 at zero. For  $H > 1$ ,  $\tau \in \mathbb{R}$ , let us define:

$$\Lambda_\psi(\tau, H) := \sum_{n \geq 1} \ell(n) n^{-\frac{1}{2} - i\tau} \psi\left(\frac{\log n}{H}\right).$$

Then, for  $\kappa \in \{1, -i, i\}$ ,  $h > 0$ ,  $t \in \mathbb{R}$ ,  $A > 0$ , we have

$$\begin{aligned} & \sup_{\tau \in [t-h, t+h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \\ & \geq \sup_{\tau \in [t-(h/2), t+(h/2)]} \Re(\kappa \Lambda_\psi(\tau, H)) + \mathcal{O}_{\varphi, A, h} \left( 1 + H^{-A} \log(2 + |t|) + \frac{e^{\mathcal{O}_\varphi(H)}}{1 + |t|} \right). \end{aligned}$$

*Proof.* By applying Proposition 3.2, we get for  $\sigma \in (1/2, 1)$ ,

$$\begin{aligned} \Re \left( \kappa \sum_{n \geq 1} \ell(n) n^{-\sigma - i\tau} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right) &= \int_{-\infty}^{\infty} \Re(\kappa \log \zeta(\sigma + i(\tau + tH^{-1}))) \varphi(t) dt \\ &+ \mathcal{O}_\varphi \left( 1 + \frac{e^{\mathcal{O}_\varphi(H)}}{1 + |\tau|} \right). \end{aligned}$$

Since  $\varphi$  is nonnegative,

$$\begin{aligned} & \int_{-\infty}^{\infty} \Re(\kappa \log \zeta(\sigma + i(\tau + tH^{-1}))) \varphi(t) dt \\ & \leq \left( \int_{-hH/2}^{hH/2} \varphi(t) dt \right) \left( \sup_{u \in [\tau - h/2, \tau + h/2]} \Re(\kappa \log(\zeta(\sigma + iu))) \right) \\ & + \int_{\mathbb{R} \setminus [-hH/2, hH/2]} \Re(\kappa \log \zeta(\sigma + i(\tau + tH^{-1}))) \varphi(t) dt. \end{aligned}$$

Now, for  $\sigma \in (1/2, 3/4)$  (say),  $(\log |\zeta(\sigma + i\tau)|)_+ = \mathcal{O}(\log(2 + |\tau|))$  (for example, from Theorem 4.11 of [Tit86]). On the other hand, from Theorem 9.2 and Theorem 9.6 (B) of [Tit86],  $|\Im \log \zeta(\sigma + i\tau)| = \mathcal{O}(\log(2 + |\tau|))$ . Hence, the positive part of the last integral is dominated by

$$\begin{aligned} & \int_{\mathbb{R} \setminus [-hH/2, hH/2]} \log(2 + |\tau| + |t|H^{-1}) \varphi(t) dt \leq \int_{\mathbb{R} \setminus [-hH/2, hH/2]} \log((2 + |\tau|)(2 + |t|)) \varphi(t) dt \\ & \leq \log(2 + |\tau|) \int_{\mathbb{R} \setminus [-hH/2, hH/2]} \varphi(t) dt + \int_{\mathbb{R} \setminus [-hH/2, hH/2]} \log(2 + |t|) \varphi(t) dt \\ & \ll_A (Hh)^{-A} \log(2 + |\tau|), \end{aligned}$$

for all  $A > 0$ . Here, in the first inequality, we used that  $H > 1$ , and in the last estimate, that  $\varphi$  is rapidly decaying at infinity.

Hence,

$$\begin{aligned} & \int_{-\infty}^{\infty} \Re(\kappa \log \zeta(\sigma + i(\tau + tH^{-1}))) \varphi(t) dt \\ & \leq (1 - \mathcal{O}_A((Hh)^{-A})) \sup_{u \in [\tau - h/2, \tau + h/2]} (\Re(\kappa \log \zeta(\sigma + iu)))_+ + \mathcal{O}_A((Hh)^{-A} \log(2 + |\tau|)) \\ & \leq \sup_{u \in [\tau - h/2, \tau + h/2]} (\Re(\kappa \log \zeta(\sigma + iu)))_+ + \mathcal{O}_A((Hh)^{-A} \log(2 + |\tau| + h)), \end{aligned}$$

and

$$\begin{aligned} & \Re \left( \kappa \sum_{n \geq 1} \ell(n) n^{-\sigma - i\tau} \widehat{\varphi} \left( \frac{\log n}{H} \right) \right) \\ & \leq \sup_{u \in [\tau - h/2, \tau + h/2]} (\Re(\kappa \log \zeta(\sigma + iu)))_+ + \mathcal{O}_{\varphi, A, h} \left( 1 + H^{-A} \log(2 + |\tau|) + \frac{e^{\mathcal{O}_{\varphi}(H)}}{1 + |\tau|} \right). \end{aligned}$$

Taking the supremum over  $\tau \in [t - h/2, t + h/2]$ , which gives

$$1 + |\tau| \gg_h 1 + |t|, \log(2 + |\tau|) \ll_h \log(2 + |t|),$$

we deduce

$$\begin{aligned} & \sup_{\tau \in [t - h/2, t + h/2]} \Re \left( \kappa \sum_{n \geq 1} \ell(n) n^{-\sigma - i\tau} \psi \left( \frac{\log n}{H} \right) \right) \\ & \leq \sup_{u \in [t - h, t + h]} (\Re(\kappa \log \zeta(\sigma + iu)))_+ + \mathcal{O}_{\varphi, A, h} \left( 1 + H^{-A} \log(2 + |t|) + \frac{e^{\mathcal{O}_{\varphi}(H)}}{1 + |t|} \right) \end{aligned}$$

When  $\sigma$  goes to  $1/2$ , the left-hand side tends to the supremum on  $[t - h/2, t + h/2]$  of  $\Lambda_{\psi}(\tau, H)$ , by uniform continuity of the sum with respect to  $\sigma$  (this sum has finitely many nonzero terms), on the interval where  $\tau$  varies. Similarly, by using local uniform continuity of  $\zeta$  far from 1, we deduce

$$\sup_{u \in [t - h, t + h]} \max(1, |\zeta(\sigma + iu)|) \xrightarrow{\sigma \rightarrow 1/2} \sup_{u \in [t - h, t + h]} \max(1, |\zeta(1/2 + iu)|).$$

Hence, taking the limit  $\sigma \rightarrow 1/2$  gives the claim of the proposition for  $\kappa = 1$ .

On the other hand, if for  $\sigma \in (1/2, 3/4)$ ,  $u_{\sigma}$  denotes a point where  $(\pm \Im(\log \zeta(\sigma + iu)))_+$  is at distance less than 1 from its supremum, then one can extract a sequence  $(\sigma_r)_{r \geq 1}$  decreasing to  $1/2$ , such that  $u_{\sigma_r}$  converges to a limit  $u^*$ . If  $(1/2) + iu^*$  is not a zero of  $\zeta$ , then by continuity of  $\log \zeta$  at  $(1/2) + iu^*$ ,

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \sup_{u \in [t - h, t + h]} (\pm \Im(\log \zeta(\sigma_r + iu)))_+ \\ & \leq \limsup_{r \rightarrow \infty} (\pm \Im(\log \zeta(\sigma_r + iu_{\sigma_r})))_+ + \mathcal{O}(1) \\ & = (\pm \Im(\log \zeta(1/2 + iu^*)))_+ + \mathcal{O}(1) \\ & \leq \sup_{u \in [t - h, t + h]} (\pm \Im(\log \zeta(1/2 + iu)))_+ + \mathcal{O}(1). \end{aligned}$$

Since the jumps of  $\Im \log \zeta$  are bounded, this estimate remains true for  $\zeta(1/2 + iu^*) = 0$ , by replacing, in the third line,  $\zeta(1/2 + iu^*)$  by  $\zeta(1/2 + iv)$  for  $v$  above or below  $u^*$ , sufficiently close to  $u^*$ . Taking the  $\limsup$  for  $r \rightarrow \infty$  then gives the claim in the case  $\kappa \in \{-i, i\}$ .  $\square$

When  $H$  satisfies some extra condition, the statement of the previous proposition can be simplified:

**Proposition 5.3.** *Let us assume the Riemann hypothesis, and let  $\varphi$  be a function satisfying the assumptions of the previous proposition. Then, there exists  $\alpha > 0$ , depending only on  $\varphi$ , such that for  $h > 0$ ,  $t \in \mathbb{R}$ ,  $(\log(3 + |t|))^{1/10} \leq H \leq \alpha \log(3 + |t|)$ ,*

$$\sup_{\tau \in [t - h, t + h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \geq \sup_{\tau \in [t - (h/2), t + (h/2)]} \Re(\kappa \Lambda_{\psi}(\tau, H)) + \mathcal{O}_{\varphi, h}(1).$$

*Proof.* From the assumption,  $H \geq (\log 3)^{1/10} > 1$ , and then we can apply the previous proposition, with  $A = 10$ . In the error term,

$$H^{-A} \log(2 + |t|) \leq (\log(3 + |t|))^{-1} \log(2 + |t|) \leq 1,$$

and if we take  $\alpha$  (depending only on  $\varphi$ ) such that  $\mathcal{O}_\varphi(H) \leq H/\alpha$ , we get

$$\frac{e^{\mathcal{O}_\varphi(H)}}{1 + |t|} \leq \frac{e^{\log(3 + |t|)}}{1 + |t|} \leq 3,$$

which implies that the error term is  $\mathcal{O}_{\varphi,h}(1)$ .  $\square$

In the proposition we have just proven, we deduce the following, by taking  $t$  uniformly at random in  $[0, T]$ :

**Proposition 5.4.** *Let us assume the Riemann hypothesis, and let  $\varphi$  satisfy the same assumptions as in the two previous propositions. Let  $h > 0$ ,  $T > 0$ ,  $H \geq (\log(3 + T))^{1/10}$  integer depending only on  $T$ , negligible with  $\log T$  when  $T$  goes to infinity, and let  $U$  be a uniform variable on  $[0, 1]$ . Then, with probability tending to 1 when  $T$  goes to infinity, we have*

$$\begin{aligned} & \sup_{\tau \in [UT-h, UT+h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \\ & \geq \sup_{k \in \{0, 1, \dots, H-1\}} \Re(\kappa \Lambda_\psi(TU - h/2 + kh/H, H)) + \mathcal{O}_{\varphi,h}(1). \end{aligned}$$

*Proof.* We have  $H \geq (\log(3 + UT))^{1/10}$ , and for  $T$  large enough,  $H \leq (\alpha/2) \log(3 + T)$ , which implies

$$\begin{aligned} \mathbb{P}[H \geq \alpha \log(3 + UT)] & \leq \mathbb{P}[(\alpha/2) \log(3 + T) \geq \alpha \log(3 + UT)] \leq \mathbb{P}[3 + T \geq (3 + UT)^2] \\ & \leq \mathbb{P}[3 + T \geq 9 + U^2 T^2] \leq \mathbb{P}[U \leq 1/\sqrt{T}] \xrightarrow{T \rightarrow \infty} 0. \end{aligned}$$

Hence, we can apply the previous proposition to  $t = TU$  with probability tending to 1 when  $T$  goes to infinity. By restricting the supremum in the right-hand side of the minorization, we are done.  $\square$

The expression of  $\Re(\kappa \Lambda_\psi)$  is a sum indexed by the powers of primes. The following result shows that with high probability, one can get rid of the powers with exponent strictly larger than 1:

**Proposition 5.5.** *With the notation of the previous proposition, and under the extra condition  $H \leq \frac{\log T}{(\log \log T)^2}$ , we have with probability tending to 1 when  $T$  goes to infinity:*

$$\begin{aligned} & \sup_{k \in \{0, 1, \dots, H-1\}} \left| \Lambda_\psi(TU - h/2 + kh/H, H) - \sum_{p \in \mathcal{P}} p^{-\frac{1}{2} - i(TU - h/2 + kh/H)} \psi \left( \frac{\log p}{H} \right) \right| \\ & = \mathcal{O}(\sqrt{\log \log T}). \end{aligned}$$

*Proof.* It is sufficient to check that with probability tending to 1,

$$\sup_{k \in \{0, 1, \dots, H-1\}} \left| \sum_{r \geq 3} \sum_{p \in \mathcal{P}} \frac{1}{r} p^{-\frac{r}{2} - ir(TU - h/2 + kh/H)} \psi \left( \frac{r \log p}{H} \right) \right| = \mathcal{O}(\sqrt{\log \log T}),$$

$$\sup_{k \in \{0, 1, \dots, H-1\}} \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right| = \mathcal{O}(\sqrt{\log \log T}).$$

The first supremum is uniformly bounded by the universal constant

$$\sum_{r \geq 3} \sum_{n \geq 2} n^{-r/2} < \infty,$$

it is then sufficient to bound the second supremum. For  $r \geq 0$  integer, the  $2r$ -th moment of the quantity inside the supremum is equal to

$$2^{-2r} \sum_{p_1, \dots, p_r, q_1, \dots, q_r \in \mathcal{P}} \prod_{j=1}^r (p_j q_j)^{-1} \psi \left( \frac{2 \log p_j}{H} \right) \psi \left( \frac{2 \log q_j}{H} \right) \int_0^1 \left( \frac{p_1 \dots p_r}{q_1 \dots q_r} \right)^{-2i(uT-h/2+kh/H)} du.$$

The sum of the terms such that

$$\prod_{j=1}^r p_j = \prod_{j=1}^r q_j$$

is equal to

$$\mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{X_p}{2p} \psi \left( \frac{2 \log p}{H} \right) \right|^{2r} \right],$$

where  $(X_p)_{p \in \mathcal{P}}$  are i.i.d., uniform on the unit circle. For any other term, the integral above between 0 and 1 is at most, in absolute value,

$$\left( T \left| \log \left( \frac{p_1 \dots p_r}{q_1 \dots q_r} \right) \right| \right)^{-1} = T^{-1} \left| \int_{p_1 \dots p_r}^{q_1 \dots q_r} \frac{dx}{x} \right|^{-1} \leq \frac{\max(p_1 \dots p_r, q_1 \dots q_r)}{T},$$

since the two bounds of the integral are two distinct integers, which implies that the length of the interval of integration is at least 1. We deduce that the sum of the terms for which  $p_1 \dots p_r \neq q_1 \dots q_r$  is at most, in absolute value:

$$2^{1-2r} T^{-1} \sum_{p_1, \dots, p_r, q_1 \dots q_r \in \mathcal{P} \cap [1, e^{AH/2}], p_1 \dots p_r < q_1 \dots q_r} (p_1 \dots p_r)^{-1},$$

if  $\psi$  is supported in  $[-A, A]$  (recall that  $0 \leq \psi \leq 1$ ). The last sum is bounded by

$$\left( \sum_{p \in \mathcal{P} \cap [1, e^{AH/2}]} p^{-1} \right)^r \left( \sum_{p \in \mathcal{P} \cap [1, e^{AH/2}]} 1 \right)^r = [\mathcal{O}(\log(2+AH))]^r \left( \mathcal{O} \left( \frac{e^{AH/2}}{1+AH} \right) \right)^r \leq B^r e^{rAH/2},$$

where  $B > 1$  is a universal constant. We then get, for  $r \geq 1$ ,

$$\mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right|^{2r} \right] \leq \frac{B^r e^{rAH/2}}{T} + \mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{X_p}{2p} \psi \left( \frac{2 \log p}{H} \right) \right|^{2r} \right].$$

By crudely bounding each term of the sum on  $p \in \mathcal{P}$  by its absolute value, we also get

$$\mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right|^{2r} \right] \leq \left( \sum_{p \in \mathcal{P} \cap [1, e^{AH/2}]} p^{-1} \right)^{2r} \leq (B' \log(2+AH))^{2r},$$

where  $B' > 1$  is universal. By summing the hyperbolic cosine series, we obtain, separating the cases  $B^r e^{rAH/2} \leq T$  and  $B^r e^{rAH/2} > T$ , for  $\lambda > 0$ ,

$$\begin{aligned} & \mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right| \right) \right] \\ & \leq \sum_{0 \leq r \leq \frac{2 \log T}{AH+2 \log B}} \frac{\lambda^{2r}}{(2r)!} + \sum_{r > \frac{2 \log T}{AH+2 \log B}} \frac{(\lambda B' \log(2+AH))^{2r}}{(2r)!} + \mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{X_p}{2p} \psi \left( \frac{2 \log p}{H} \right) \right| \right) \right]. \end{aligned}$$

The first sum is bounded by  $\cosh \lambda \leq e^\lambda$ . Now, by looking at the ratio between two consecutive terms and by bounding the first term with the Stirling formula, we deduce, for all  $u > 0$ ,

$$\sum_{r \geq 2u} \frac{u^{2r}}{(2r)!} \ll 1.$$

Hence, the second sum above is dominated by 1, provided that

$$\frac{2 \log T}{AH + 2 \log B} \geq 2B' \lambda \log(2+AH),$$

i.e.

$$B' \lambda (AH + 2 \log B) \log(2+AH) \leq \log T.$$

Now, since we assume that  $H \leq \frac{\log T}{(\log \log T)^2}$ , we check that this condition is satisfied for  $T$  large enough (depending on  $\varphi$ ), if  $\lambda \ll \sqrt{\log \log T}$ . Finally, from the inequality

$$\cosh |z| \leq e^{|z|} \leq e^{|\Re(z)|+|\Im(z)|} \leq e^{2 \sup(|\Re(z)|, |\Im(z)|)} \leq e^{2\Re(z)} + e^{-2\Re(z)} + e^{2\Im(z)} + e^{-2\Im(z)},$$

the rotation invariance and the symmetry of the law of  $X_p$ , we get

$$\begin{aligned} & \mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{X_p}{2p} \psi \left( \frac{2 \log p}{H} \right) \right| \right) \right] \leq 4 \mathbb{E} \left[ \exp \left( 2\lambda \Re \sum_{p \in \mathcal{P}} \frac{X_p}{2p} \psi \left( \frac{2 \log p}{H} \right) \right) \right] \\ & = 4 \prod_{p \in \mathcal{P} \cap [1, e^{AH/2}]} \mathbb{E} \left[ e^{\frac{\lambda \psi(2H^{-1} \log p)}{p} \Re X_p} \right] = 4 \prod_{p \in \mathcal{P} \cap [1, e^{AH/2}]} \mathbb{E} \left[ \cosh \left( \frac{\lambda \psi(2H^{-1} \log p)}{p} \Re X_p \right) \right] \\ & \leq 4 \prod_{p \in \mathcal{P} \cap [1, e^{AH/2}]} \cosh(\lambda/p) \leq 4 \prod_{p \in \mathcal{P} \cap [1, e^{AH/2}]} e^{\lambda^2/2p^2} \leq 4e^{\lambda^2 \sum_{n \geq 2} n^{-2}/2} \leq 4e^{\lambda^2}. \end{aligned}$$

Hence, for  $0 \leq \lambda \ll \sqrt{\log \log T}$  and  $T$  large enough depending on  $\varphi$ ,

$$\mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right| \right) \right] \ll e^{\lambda^2}.$$

and then

$$\mathbb{E} \left[ \exp \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right| \right) \right] \ll e^{\lambda^2}.$$

The probability that the sum inside the exponential is larger than  $2\sqrt{\log \log T}$  is then dominated by

$$e^{-2\lambda\sqrt{\log \log T} + \lambda^2} = e^{-\log \log T} = \frac{1}{\log T},$$

by taking  $\lambda = \sqrt{\log \log T}$ . A union bound on  $k$  gives

$$\mathbb{P} \left[ \sup_{k \in \{0, 1, \dots, H-1\}} \left| \sum_{p \in \mathcal{P}} \frac{1}{2} p^{-1-2i(UT-h/2+kh/H)} \psi \left( \frac{2 \log p}{H} \right) \right| \geq 2\sqrt{\log \log T} \right] = \mathcal{O} \left( \frac{H}{\log T} \right),$$

which tends to zero when  $T$  goes to infinity.  $\square$

In the next result, we show that we can replace the smooth cutoff by a sharp cutoff, and then get rid of the function  $\varphi$ :

**Proposition 5.6.** *Under the same condition as in the previous proposition, we have with probability tending to 1 when  $T$  goes to infinity:*

$$\sup_{k \in \{0, 1, \dots, H-1\}} \left| \sum_{p \in \mathcal{P}} p^{-\frac{1}{2}-i(TU-h/2+kh/H)} \left( \mathbf{1}_{p \leq e^H} - \psi \left( \frac{\log p}{H} \right) \right) \right| = \mathcal{O}_\varphi(\sqrt{\log \log T}).$$

*Proof.* As above, we first compute the moment of order  $2r$  of the quantity inside the supremum, and we get:

$$\sum_{p_1, \dots, p_r, q_1, \dots, q_r \in \mathcal{P}} \prod_{j=1}^r (p_j q_j)^{-1/2} \chi \left( \frac{\log p_j}{H} \right) \chi \left( \frac{\log q_j}{H} \right) \int_0^1 \left( \frac{p_1 \dots p_r}{q_1 \dots q_r} \right)^{-i(uT-h/2+kh/H)} du,$$

where  $\chi(x) := \mathbf{1}_{|x| \leq 1} - \psi(x)$ . The sum of the terms with  $p_1 \dots p_r = q_1 \dots q_r$  is equal to

$$\mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right|^{2r} \right].$$

By majorizing the oscillating integral between 0 and 1 as in the previous proof, we get a majorization, in absolute value, of the sum of the terms with  $p_1 \dots p_r \neq q_1 \dots q_r$ , by

$$4T^{-1} \sum_{p_1, \dots, p_r, q_1 \dots q_r \in \mathcal{P} \cap [1, e^{A'H}], p_1 \dots p_r < q_1 \dots q_r} (p_1 \dots p_r)^{-1/2} (q_1 \dots q_r)^{1/2}$$

where  $A' = \max(A, 1)$  (recall that  $-1 \leq \chi \leq 1$  and that  $\chi$  is supported in  $[-A', A']$ ). Now, for  $a \in \{-1/2, 1/2\}$ , the crude bound

$$\sum_{p \in \mathcal{P} \cap [1, e^{A'H}]} p^a \leq \sum_{n=1}^{\lfloor e^{A'H} \rfloor} n^a \leq \int_0^{e^{A'H}+1} x^a dx \leq \frac{(2e^{A'H})^{a+1}}{a+1} \leq 3e^{(a+1)A'H},$$

used  $r$  times for  $a = 1/2$  and  $r$  times for  $a = -1/2$ , gives

$$\begin{aligned} & \mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} p^{-\frac{1}{2}-i(TU-h/2+kh/H)} \left( \mathbf{1}_{p \leq e^H} - \psi \left( \frac{\log p}{H} \right) \right) \right|^{2r} \right] \\ & \leq \mathbb{E} \left[ \left| \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right|^{2r} \right] + \frac{4 \cdot 3^{2r} e^{2rA'H}}{T}. \end{aligned}$$

Now, if we write

$$\Delta = \min \left( \left| \sum_{p \in \mathcal{P}} p^{-\frac{1}{2}-i(TU-h/2+kh/H)} \left( \mathbf{1}_{p \leq e^H} - \psi \left( \frac{\log p}{H} \right) \right) \right|, \log \log T \right),$$

we have obviously  $\mathbb{E}[\Delta^{2r}] \leq (\log \log T)^{2r}$ , and then, by expanding the hyperbolic cosine, we get, for all  $\lambda > 0$ ,

$$\begin{aligned} \mathbb{E}[\cosh(\lambda \Delta)] &\leq \mathbb{E} \left[ \cosh \left| \lambda \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right| \right] + \sum_{0 \leq r \leq \log T / (2A'H)} \frac{4 \cdot (3\lambda)^{2r}}{(2r)!} \\ &\quad + \sum_{r > \log T / (2A'H)} \frac{(\lambda \log \log T)^{2r}}{(2r)!} \\ &= \mathbb{E} \left[ \cosh \left| \lambda \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right| \right] + \mathcal{O}(e^{3\lambda}), \end{aligned}$$

provided that

$$\frac{\log T}{2A'H} \geq 2\lambda \log \log T.$$

For  $\lambda \ll \sqrt{\log \log T}$ , this inequality is true for  $T$  large enough since we assume  $H \leq \frac{\log T}{(\log \log T)^2}$ . As in the previous proof, we get

$$\begin{aligned} \mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right| \right) \right] &\leq 4 \mathbb{E} \left[ \exp \left( 2\lambda \Re \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right) \right] \\ &= 4 \prod_{p \in \mathcal{P} \cap [1, e^{A'H}]} \mathbb{E} \left[ e^{\frac{2\lambda \chi(H^{-1} \log p)}{\sqrt{p}} \Re X_p} \right] = 4 \prod_{p \in \mathcal{P} \cap [1, e^{A'H}]} \mathbb{E} \left[ \cosh \left( \frac{2\lambda \chi(H^{-1} \log p)}{\sqrt{p}} \Re X_p \right) \right] \\ &\leq 4 \prod_{p \in \mathcal{P} \cap [1, e^{A'H}]} \cosh(2\lambda \chi(H^{-1} \log p) / \sqrt{p}) \leq 4 \prod_{p \in \mathcal{P} \cap [1, e^{A'H}]} e^{2\lambda^2 \chi^2(H^{-1} \log p) / p}. \end{aligned}$$

Since  $\chi$  is smooth on  $[0, 1]$  and equal to 0 at 0, we have

$$|\chi(x)| \leq \int_0^x |\chi'(y)| dy \leq x \sup_{[0,1]} |\chi'| \ll_{\varphi} x$$

for  $x \in [0, 1]$ . Of course, this estimate remains true for  $x > 1$  since  $|\chi| \leq 1$ . Hence

$$\sum_{p \in \mathcal{P} \cap [1, e^{A'H}]} \frac{\chi^2(H^{-1} \log p)}{p} \ll_{\varphi} H^{-2} \sum_{p \in \mathcal{P} \cap [1, e^{A'H}]} \frac{\log^2 p}{p}.$$

If  $\pi$  denotes the prime counting function, we get for  $t > 1$ , by using the prime number theorem at the third line (a weak form is sufficient here):

$$\begin{aligned} \sum_{p \in \mathcal{P} \cap [1, t]} \frac{\log^2 p}{p} &= \int_{[1, t]} \frac{\log^2 x}{x} d\pi(x) \\ &= \left[ \frac{\log^2 x}{x} \pi(x) \right]_1^t - \int_1^t \frac{2 \log x - \log^2 x}{x^2} \pi(x) dx \\ &\ll \log t + \int_1^t \frac{1 + \log x}{x} dx \ll \log^2 t. \end{aligned}$$

This estimates gives

$$\sum_{p \in \mathcal{P} \cap [1, e^{A'H}]} \frac{\chi^2(H^{-1} \log p)}{p} \ll_{\varphi} 1,$$

$$\mathbb{E} \left[ \cosh \left( \lambda \left| \sum_{p \in \mathcal{P}} \frac{X_p}{\sqrt{p}} \chi \left( \frac{\log p}{H} \right) \right| \right) \right] \ll e^{\mathcal{O}_\varphi(\lambda^2)}$$

and then

$$\mathbb{E}[e^{\lambda \Delta}] \ll \mathbb{E}[\cosh(\lambda \Delta)] \ll e^{\mathcal{O}_\varphi(\lambda^2)}.$$

By taking  $\lambda = \sqrt{\log \log T}$ , we get, for  $C > 0$ ,

$$\mathbb{P}[\Delta \geq C \sqrt{\log \log T}] \ll e^{-(C - \mathcal{O}_\varphi(1)) \log \log T} \ll \frac{1}{\log T},$$

if  $C$  is large enough depending only on  $\varphi$ . Since  $C \sqrt{\log \log T} \leq \log \log T$  for  $T$  large enough depending only on  $\varphi$ , we deduce, under these conditions,

$$\mathbb{P} \left[ \left| \sum_{p \in \mathcal{P}} p^{-\frac{1}{2} - i(TU - h/2 + kh/H)} \left( \mathbb{1}_{p \leq e^H} - \psi \left( \frac{\log p}{H} \right) \right) \right| \geq \mathcal{O}_\varphi(\sqrt{\log \log T}) \right] \ll \frac{1}{\log T}$$

Taking the union bound on  $k \in \{0, 1, \dots, H-1\}$  gives the desired result, since  $H = o(\log T)$  for  $T \rightarrow \infty$ .  $\square$

We have now all the elements we need to show Proposition 5.1. We first arbitrarily fix a function  $\varphi$  satisfying the assumptions of Proposition 5.2, which is possible by Proposition 3.4. Then, we combine Propositions 5.4, 5.5 and 5.6.

## 6 Comparison with Gaussian variables

From now, we fix the following quantities:  $h > 0$ ,  $T > 10$ ,  $\delta \in (0, 1/2)$ ,  $K \geq 2$  integer,  $H := \lfloor (\log T)^{1-\delta} \rfloor$ . For  $T$  large enough depending on  $\delta$ , Proposition 5.1 applies, since  $(\log(3+T))^{1/10} \leq H \leq \frac{\log T}{(\log \log T)^2}$ . We then define, for  $m \in \{0, 1, \dots, K-1\}$ ,  $k \in \{0, 1, \dots, H-1\}$ :

$$S(k, m) := \Re \left( \kappa \sum_{p \in \mathcal{P} \cap (e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}]} p^{-\frac{1}{2} - i(TU - h/2 + kh/H)} \right).$$

We now see that the main term in the right-hand side of the estimate in Proposition 5.1 is (up to  $\mathcal{O}(1)$  because of the term indexed by  $p = 2 \leq e^{e^0}$ ) equal to the supremum, for  $k \in \{0, 1, \dots, H-1\}$ , of the sum of  $S(k, m)$  for  $m \in \{0, 1, \dots, K-1\}$ . Hence, if we show that with high probability, there exists  $k \in \{0, 1, \dots, H-1\}$  such that all the values of  $S(k, m)$  ( $m \in \{0, 1, \dots, K-1\}$ ) are large, then we will deduce that with high probability, the supremums involved in Proposition 5.1 are also large. This will give a lower bound for the supremum of  $\Re(\kappa \log \zeta)$  on the segment  $[1/2 + i(UT - h), 1/2 + i(UT + h)]$ .

Note that the random variable  $S(k, m)$  implicitly depends on  $T$ ,  $\delta$  (which together give  $H$ ),  $K$ ,  $h$  and  $\kappa$ . For technical reasons, we will replace  $S(k, m)$  by a truncated version, defined as follows:

$$S_0(k, m) := \min((\log T)^{\delta/3}, \max(-(\log T)^{\delta/3}, S(k, m))).$$

We will show that in a sense which is made precise, the variables  $S_0(k, m)$  for  $0 \leq k \leq H-1$ ,  $0 \leq m \leq K-1$  are not far from being the components of a Gaussian vector with a similar covariance structure.

This comparison with Gaussian variables will be done in two steps.

In the first step, we observe that the random phases  $(p^{-iUT})_{p \in \mathcal{P}}$  tend in law to i.i.d. uniform random variables  $(X_p)_{p \in \mathcal{P}}$  on the unit circle, in the sense of the finite-dimensional



marginals. It is then natural to compare the variables  $S(k, m)$  and  $S_0(k, m)$  with the variables  $V(k, m)$  defined by

$$V(k, m) := \Re \left( \kappa \sum_{p \in \mathcal{P} \cap (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}} X_p p^{-\frac{1}{2} - i(-h/2 + kh/H)} \right).$$

Indeed, we will show that the joint Fourier-Laplace transforms of  $S_0(k, m)$  and  $V(k, m)$  are close to each other, when they are taken at points whose modulus is not too large. For this purpose, we need to get a comparison between the variables  $(p^{-iUT})_{p \in \mathcal{P}}$  and  $(X_p)_{p \in \mathcal{P}}$  which goes beyond the finite-dimensional marginals, and in particular, in moment computations, it is essential to deal with sums only involving primes which are much smaller than  $T$  (indeed,  $e^H \leq e^{(\log T)^{1-\delta}} = T^{o(1)}$ ).

The second step consists in a comparison between the variables  $V(k, m)$  and the components of a Gaussian vector. In this step, we will use the independence of the variables  $(X_p)_{p \in \mathcal{P}}$  in a crucial way. Indeed, if we remove the variables  $V(k, 0)$  which involve the small primes, the variables  $V(k, m)$ ,  $1 \leq m \leq K-1$  involve sums of many independent variables with small variances. Comparison between similar sums and Gaussian variables have also been done, for example, by Arguin, Belius and Harper [ABH15] in their study of the supremum of randomized versions of  $\zeta$ , or earlier, by Kowalski and Nikeghbali [KN12].

Here is the main result obtained in the first step:

**Proposition 6.1.** *For all  $\lambda_0, \dots, \lambda_{K-1}, \mu_0, \dots, \mu_{K-1}$ , complex with modulus at most  $(\log T)^{\delta/100}$ , and for all  $k, \ell \in \{0, 1, \dots, H-1\}$ , we have*

$$\mathbb{E} \left[ \exp \left( \sum_{m=0}^{K-1} (\lambda_m S_0(k, m) + \mu_m S_0(\ell, m)) \right) \right] = \mathbb{E} \left[ \exp \left( \sum_{m=0}^{K-1} (\lambda_m V(k, m) + \mu_m V(\ell, m)) \right) \right] + \mathcal{O} \left( \exp(-(\log T)^{\delta/4}) \right),$$

if  $T$  is large enough depending only on  $\delta$  and  $K$ .

*Proof.* Let  $M$  be the maximum of the moduli of the  $\lambda_m$ 's and the  $\mu_m$ 's. For  $p \in \mathcal{P} \cap [3, e^H]$ , let us denote  $\alpha_p := \kappa \lambda_m$ ,  $\beta_p := \kappa \mu_m$ ,  $\gamma_p := \bar{\kappa} \lambda_m$ ,  $\delta_p := \bar{\kappa} \mu_m$  if  $p \in (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}$ . If we denote

$$\mathcal{S} := \sum_{m=0}^{K-1} (\lambda_m S(k, m) + \mu_m S(\ell, m)),$$

we have

$$\mathcal{S} = \frac{1}{2} \sum_{p \in \mathcal{P} \cap [3, e^H]} p^{-1/2} \left( p^{-i(TU-h/2)} (\alpha_p p^{-ikh/H} + \beta_p p^{-ilh/H}) + p^{i(TU-h/2)} (\gamma_p p^{ikh/H} + \delta_p p^{ilh/H}) \right).$$

Expanding the  $r$ -th power, for  $r \geq 1$  integer, gives

$$\begin{aligned} \mathbb{E}[\mathcal{S}^r] &= 2^{-r} \sum_{p_1, \dots, p_r \in \mathcal{P} \cap [3, e^H]} P^{-1/2} \sum_{A \amalg B \amalg C \amalg D = \{1, \dots, r\}} \alpha_{P_A} P_A^{-i(-h/2+kh/H)} \dots \\ &\quad \dots \times \beta_{P_B} P_B^{-i(-h/2+\ell h/H)} \gamma_{P_C} P_C^{i(-h/2+kh/H)} \delta_{P_D} P_D^{i(-h/2+\ell h/H)} \int_0^1 \left( \frac{P_C P_D}{P_A P_B} \right)^{iT u} du, \end{aligned}$$

where, for  $X \in \{A, B, C, D\}$ ,

$$P := \prod_{j=1}^r p_j, \quad P_X := \prod_{j \in X} p_j,$$

$$\alpha_{P_X} := \prod_{j \in X} \alpha_{p_j}, \quad \beta_{P_X} := \prod_{j \in X} \beta_{p_j},$$

$$\gamma_{P_X} := \prod_{j \in X} \gamma_{p_j}, \quad \delta_{P_X} := \prod_{j \in X} \delta_{p_j}.$$

If we add the terms for which  $P_A P_B = P_C P_D$ , we get exactly  $\mathbb{E}[\mathcal{V}^r]$ , where

$$\mathcal{V} := \sum_{m=0}^{K-1} (\lambda_m V(k, m) + \mu_m V(\ell, m)).$$

Let us now bound the other terms. We get

$$\left| \int_0^1 \left( \frac{P_C P_D}{P_A P_B} \right)^{iTu} du \right| \leq 2 \left( T \left| \log \left( \frac{P_C P_D}{P_A P_B} \right) \right| \right)^{-1} \leq 2T^{-1} \max(P_A P_B, P_C P_D) \leq \frac{2P}{T}.$$

Since

$$|\alpha_{P_A} \beta_{P_B} \alpha_{P_C} \beta_{P_D}| \leq M^{|A|+|B|+|C|+|D|} \leq M^r,$$

we get

$$\begin{aligned} |\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{V}^r]| &\leq 2^{-r} \sum_{p_1, \dots, p_r \in \mathcal{P} \cap [3, e^H]} P^{-1/2} (4^r) (M^r) (2P/T) \\ &\leq 2(2M)^r T^{-1} \sum_{p_1, \dots, p_r \in \mathcal{P} \cap [3, e^H]} P^{1/2} \\ &\leq 2(2M)^r T^{-1} \left( \sum_{p \in \mathcal{P} \cap [3, e^H]} p^{1/2} \right)^r \leq \frac{2(2M)^r e^{3Hr/2}}{T}. \end{aligned}$$

Now, let us compare the moment of  $\mathcal{S}$  with the moment of

$$\mathcal{S}_0 := \sum_{m=0}^{K-1} (\lambda_m S_0(k, m) + \mu_m S_0(\ell, m)),$$

where the variables  $S(k, m)$  and  $S(\ell, m)$  are truncated. We have

$$\begin{aligned} |\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{S}_0^r]| &\leq \mathbb{E}[|\mathcal{S}^r - \mathcal{S}_0^r| \mathbf{1}_{\mathcal{S} \neq \mathcal{S}_0}] \leq \mathbb{E}[(|\mathcal{S}|^r + |\mathcal{S}_0|^r) \mathbf{1}_{\mathcal{S} \neq \mathcal{S}_0}] \\ &\leq \sum_{m=0}^{K-1} \mathbb{E} \left[ (|\mathcal{S}|^r + |\mathcal{S}_0|^r) (\mathbf{1}_{|S(k, m)| \geq (\log T)^{\delta/3}} + \mathbf{1}_{|S(\ell, m)| \geq (\log T)^{\delta/3}}) \right] \\ &\leq \sum_{m=0}^{K-1} \mathbb{E} \left[ \left( \left( M \sum_{m'=0}^{K-1} (|S(k, m')| + |S(\ell, m')|) \right)^r + \left( M \sum_{m'=0}^{K-1} (|S_0(k, m')| + |S_0(\ell, m')|) \right)^r \right) \dots \right. \\ &\quad \left. \dots \times (\mathbf{1}_{|S(k, m)| \geq (\log T)^{\delta/3}} + \mathbf{1}_{|S(\ell, m)| \geq (\log T)^{\delta/3}}) \right] \\ &\leq M^r (2K)^{r-1} \sum_{0 \leq m, m' \leq K-1} \mathbb{E} \left[ (|S(k, m')|^r + |S(\ell, m')|^r + |S_0(k, m')|^r + |S_0(\ell, m')|^r) \dots \right. \\ &\quad \left. \dots \times (\mathbf{1}_{|S(k, m)| \geq (\log T)^{\delta/3}} + \mathbf{1}_{|S(\ell, m)| \geq (\log T)^{\delta/3}}) \right] \\ &\leq 2M^r (2K)^{r-1} \sum_{0 \leq m, m' \leq K-1} \mathbb{E} \left[ (|S(k, m')|^r + |S(\ell, m')|^r) (\mathbf{1}_{|S(k, m)| \geq (\log T)^{\delta/3}} + \mathbf{1}_{|S(\ell, m)| \geq (\log T)^{\delta/3}}) \right] \end{aligned}$$

$$= 2M^r (2K)^{r-1} \sum_{j,j' \in \{1,2\}} \sum_{0 \leq m, m' \leq K-1} \mathbb{E}[\mathbb{1}_{|S(v_j, m)| \geq (\log T)^{\delta/3}} |S(v_{j'}, m')|^r]$$

where  $v_1 := k$ ,  $v_2 := \ell$ .

Hence, for  $w \geq 1$  integer,

$$\begin{aligned} |\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{S}_0^r]| &\leq 2(\log T)^{-w\delta/3} M^r (2K)^{r-1} \sum_{j,j' \in \{1,2\}} \sum_{0 \leq m, m' \leq K-1} \mathbb{E}[|S(v_j, m)|^w |S(v_{j'}, m')|^r] \\ &\leq 2(\log T)^{-w\delta/3} M^r (2K)^{r-1} \sum_{j,j' \in \{1,2\}} \sum_{0 \leq m, m' \leq K-1} \mathbb{E}[(S(v_j, m))^{2w}]^{1/2} \mathbb{E}[(S(v_{j'}, m'))^{2r}]^{1/2}. \end{aligned}$$

Now,  $\mathbb{E}[(S(k, m))^{2r}]$  corresponds to  $\mathbb{E}[\mathcal{S}^{2r}]$  in the case where  $\lambda_m = 1$ , and all the other  $\lambda_j$ 's and  $\mu_j$ 's are zero (and then  $M = 1$ ). The computation above then implies:

$$\mathbb{E}[(S(k, m))^{2r}] = \mathbb{E}[(V(k, m))^{2r}] + \mathcal{O}\left(\frac{2^{2r} e^{3Hr}}{T}\right).$$

where

$$\begin{aligned} \mathbb{E}[(V(k, m))^{2r}] &\leq \mathbb{E}\left[\left|\sum_{p \in \mathcal{P} \cap (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}} \frac{X_p}{\sqrt{p}}\right|^{2r}\right] \\ &= \sum_{p_1, \dots, p_r, q_1, \dots, q_r \in \mathcal{P} \cap (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}} \frac{\mathbb{1}_{p_1 \dots p_r = q_1 \dots q_r}}{p_1 \dots p_r} \\ &\leq r! \sum_{p_1, \dots, p_r \in (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}} \frac{1}{p_1 \dots p_r} \\ &\leq r! \left(\sum_{p \in [3, e^H]} \frac{1}{p}\right)^r = r! (\log H + \mathcal{O}(1))^r \leq r! (\log T)^{r\delta/6}, \end{aligned}$$

if  $T$  is large enough depending on  $\delta$ . We deduce:

$$\begin{aligned} |\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{S}_0^r]| &\ll (\log T)^{-w\delta/3} M^r (2K)^{r-1} K^2 \left( (w!)^{1/2} (\log T)^{w\delta/12} + \frac{2^w e^{3Hw/2}}{\sqrt{T}} \right) \left( (r!)^{1/2} (\log T)^{r\delta/12} + \frac{2^r e^{3Hr/2}}{\sqrt{T}} \right) \\ &\ll (2M)^r K^{r+1} \left( (w!)^{1/2} (\log T)^{-w\delta/4} + \frac{2^w e^{3Hw/2}}{\sqrt{T}} \right) \left( (r!)^{1/2} (\log T)^{r\delta/12} + \frac{2^r e^{3Hr/2}}{\sqrt{T}} \right). \end{aligned}$$

We now take  $w = \lfloor (\log T)^{\delta/3} \rfloor$ , which implies

$$\begin{aligned} (w!)^{1/2} (\log T)^{-w\delta/4} &\leq [w^{1/2} (\log T)^{-\delta/4}]^w \leq [(\log T)^{-\delta/12}]^w \\ &\leq \exp\left(-(\delta/12)(\log \log T) \lfloor (\log T)^{\delta/3} \rfloor\right) \\ &\leq \exp(-(\log T)^{\delta/3}), \end{aligned}$$

whereas

$$\frac{2^w e^{3Hw/2}}{\sqrt{T}} \leq T^{-1/2} \exp\left((\log T)^{\delta/3} (\log 2) + 3(\log T)^{1-(2\delta/3)/2}\right) \leq T^{-0.49},$$

if  $T$  is large enough depending on  $\delta$ . Hence,

$$|\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{S}_0^r]| \ll (2M)^r K^{r+1} \exp(-(\log T)^{\delta/3}) \left( (r!)^{1/2} (\log T)^{r\delta/12} + \frac{2^r e^{3Hr/2}}{\sqrt{T}} \right).$$

Combining with the estimate of  $\mathbb{E}[\mathcal{S}^r] - \mathbb{E}[\mathcal{V}^r]$ , we get

$$\begin{aligned} & |\mathbb{E}[\mathcal{S}_0^r] - \mathbb{E}[\mathcal{V}^r]| \\ & \ll (2M)^r K^{r+1} \exp(-(\log T)^{\delta/3}) \left( (r!)^{1/2} (\log T)^{r\delta/12} + \frac{2^r e^{3Hr/2}}{\sqrt{T}} \right) + \frac{(2M)^r e^{3Hr/2}}{T}. \end{aligned}$$

Now, under the assumption of the proposition,  $M \leq (\log T)^{\delta/100}$ . If  $r \leq (\log T)^{\delta/4}$ ,

$$\begin{aligned} & (2M)^r K^{r+1} \exp(-(\log T)^{\delta/3}) (r!)^{-1/2} (\log T)^{r\delta/12} \\ & \leq \left( 2K^2 (\log T)^{\delta(\frac{1}{100} + \frac{1}{12})} \right)^r \exp(-(\log T)^{\delta/3}) \\ & \leq \exp \left( (\log T)^{\delta/4} [\log(2K^2) + (7\delta/75) \log \log T] - (\log T)^{\delta/3} \right) \\ & \leq \exp \left( -\frac{1}{2} (\log T)^{\delta/3} \right) \end{aligned}$$

if  $T$  is large enough depending on  $\delta$  and  $K$ . If  $r \geq (\log T)^{\delta/4}$ ,

$$\begin{aligned} & (2M)^r K^{r+1} \exp(-(\log T)^{\delta/3}) (r!)^{-1/2} (\log T)^{r\delta/12} \\ & \leq \left( 2K^2 (\log T)^{\delta(\frac{1}{100} + \frac{1}{12})} \right)^r \exp(-(\log T)^{\delta/3}) (e/r)^{r/2} \\ & \leq \left( 2K^2 (\log T)^{7\delta/75} e^{1/2} (\log T)^{-\delta/8} \right)^r \exp(-(\log T)^{\delta/3}) \\ & \leq \left( 2K^2 e^{1/2} (\log T)^{-19\delta/600} \right)^r \exp(-(\log T)^{\delta/3}) \\ & \leq \exp(-(\log T)^{\delta/3}) \end{aligned}$$

for  $T$  large enough depending on  $\delta$  and  $K$ . Under the same assumptions, for  $r \leq (\log T)^{\delta/2}$ ,

$$\begin{aligned} & (2M)^r K^{r+1} \exp(-(\log T)^{\delta/3}) \frac{2^r e^{3Hr/2}}{r! \sqrt{T}} \\ & \leq (2M)^r K^{2r} \frac{2^r e^{3Hr/2}}{\sqrt{T}} \\ & \leq T^{-1/2} \exp \left( (\log T)^{\delta/2} \log(4K^2 (\log T)^{\delta/100}) + (3/2) (\log T)^{1-(\delta/2)} \right) \leq T^{-0.49} \end{aligned}$$

and

$$\frac{(2M)^r e^{3Hr/2}}{Tr!} \leq T^{-1} \exp \left( (\log T)^{\delta/2} \log(2M) + (3/2) (\log T)^{1-(\delta/2)} \right) \leq T^{-0.99}.$$

All these estimates imply that, for all  $r \leq (\log T)^{\delta/2}$ ,  $T$  large enough depending on  $\delta$  and  $K$ ,

$$\frac{1}{r!} |\mathbb{E}[\mathcal{S}_0^r] - \mathbb{E}[\mathcal{V}^r]| \ll \exp \left( -\frac{1}{2} (\log T)^{\delta/3} \right).$$

On the other hand, by the truncation,

$$|\mathcal{S}_0| \leq 2KM (\log T)^{\delta/3} \leq 2K (\log T)^{\delta(\frac{1}{3} + \frac{1}{100})} \leq (\log T)^{0.35\delta},$$

for  $T$  large enough depending on  $\delta$  and  $K$ , and then for  $r \geq (\log T)^{\delta/2}$ ,

$$\frac{1}{r!} \mathbb{E}[|\mathcal{S}_0|^r] \leq (e/r)^r (\log T)^{0.35r\delta} \leq \left( e(\log T)^{-\delta/2} (\log T)^{0.35\delta} \right)^r \leq e^{-r},$$

whereas, by using the previous estimate of  $\mathbb{E}[(V(k, m))^{2r}]$ ,

$$\begin{aligned} \frac{1}{r!} \mathbb{E}[|\mathcal{V}|^r] &= \frac{1}{r!} \|\mathcal{V}\|_r^r \leq \frac{1}{r!} \left( M \sum_{m=0}^{K-1} (\|V(k, m)\|_r + \|V(\ell, m)\|_r) \right)^r \\ &\leq \frac{1}{r!} \left( M \sum_{m=0}^{K-1} (\|V(k, m)\|_{2r} + \|V(\ell, m)\|_{2r}) \right)^r \\ &\leq \frac{1}{r!} \left( 2KM(r!)^{1/2r} (\log T)^{\delta/12} \right)^r \\ &\leq (r!)^{-1/2} (2KM(\log T)^{\delta/12})^r \\ &\leq (e/r)^{r/2} (2KM(\log T)^{\delta/12})^r \\ &\leq \left( 2e^{1/2} (\log T)^{-\delta/4} K (\log T)^{\delta/100} (\log T)^{\delta/12} \right)^r \\ &\leq e^{-r}. \end{aligned}$$

Expanding the exponential, we deduce

$$\begin{aligned} |\mathbb{E}[\exp(\mathcal{S}_0)] - \mathbb{E}[\exp(\mathcal{V})]| &\leq \sum_{r \geq 1} \frac{1}{r!} |\mathbb{E}[\mathcal{S}_0^r] - \mathbb{E}[\mathcal{V}^r]| \\ &\ll \sum_{1 \leq r \leq (\log T)^{\delta/2}} \exp\left(-\frac{1}{2}(\log T)^{\delta/3}\right) + \sum_{r > (\log T)^{\delta/2}} e^{-r} \\ &\ll (\log T)^{\delta/2} \exp\left(-\frac{1}{2}(\log T)^{\delta/3}\right) + e^{-(\log T)^{\delta/2}} \\ &\ll \exp\left(-(\log T)^{\delta/4}\right), \end{aligned}$$

when  $T$  is large enough depending on  $\delta$  and  $K$ . □

The next step consists in comparing the family  $(V(k, m))_{k \in \{0, \dots, H-1\}, m \in \{0, 1, \dots, K-1\}}$  with a Gaussian family with a close covariance structure. We have

$$V(k, m) = \Re \sum_{n \in \mathbb{N} \cap (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}] } Y_n n^{-ikh/H},$$

where

$$Y_n := \mathbb{1}_{n \in \mathcal{P}} \kappa X_n n^{-1/2+ih/2}.$$

The covariance matrix of  $Y_n$  is 0 if  $n$  is not prime and  $I_2/(2n)$  if  $n$  is prime. If we "average the variance" with the prime number theorem, we get  $I_2/(2n \log n)$ . Moreover, if we replace the sum by an integral, this leads us to compare  $V(k, m)$  with the Gaussian variable:

$$G(k, m) := \Re \int_{e^{e^m \log H/K}}^{e^{e^{(m+1) \log H/K}}} \frac{t^{-ikh/H}}{\sqrt{2t \log t}} dW_t,$$

where  $W_t := W_t^{(1)} + iW_t^{(2)}$ ,  $W^{(1)}$  and  $W^{(2)}$  being two independent standard Brownian motions. We have the following result:

**Proposition 6.2.** For all  $\lambda_1, \dots, \lambda_{K-1}, \mu_1, \dots, \mu_{K-1}$ , complex with modulus at most  $(\log T)^{1/20K}$ , and for all  $k, \ell \in \{0, 1, \dots, H-1\}$ , we have

$$\mathbb{E} \left[ \exp \left( \sum_{m=1}^{K-1} (\lambda_m V(k, m) + \mu_m V(\ell, m)) \right) \right] = \mathbb{E} \left[ \exp \left( \sum_{m=1}^{K-1} (\lambda_m G(k, m) + \mu_m G(\ell, m)) \right) \right] + \mathcal{O}_h \left( \exp(-(\log T)^{1/10K}) \right),$$

if  $T$  is large enough depending only on  $\delta$  and  $K$ .

**Remark 6.3.** The proposition is not true if we add terms involving  $V(k, 0)$  and  $V(\ell, 0)$ , since the small primes give variables which are not close to Gaussian ones.

*Proof.* We have, using the independence of the  $Y_n$ 's:

$$\begin{aligned} & \mathbb{E} [\exp (\lambda_m V(k, m) + \mu_m V(\ell, m))] \\ &= \prod_{n \in \mathbb{N} \cap (e^{m \log H/K}, e^{(m+1) \log H/K}]} \mathbb{E} [e^{\lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H})}]. \end{aligned}$$

Now, we have, from the fact that  $Y_n$  is rotationally invariant:

$$\mathbb{E} [\lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H})] = 0,$$

$$\begin{aligned} & \mathbb{E} \left[ \left( \lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H}) \right)^2 \right] \\ &= \frac{1}{4} \mathbb{E} \left[ \left( Y_n (\lambda_m n^{-ikh/H} + \mu_m n^{-i\ell h/H}) + \overline{Y_n} (\lambda_m n^{ikh/H} + \mu_m n^{i\ell h/H}) \right)^2 \right] \\ &= \frac{1}{2} (\lambda_m n^{-ikh/H} + \mu_m n^{-i\ell h/H}) (\lambda_m n^{ikh/H} + \mu_m n^{i\ell h/H}) \mathbb{E}[|Y_n|^2] \\ &= \frac{\mathbb{1}_{n \in \mathcal{P}}}{2n} (\lambda_m n^{-ikh/H} + \mu_m n^{-i\ell h/H}) (\lambda_m n^{ikh/H} + \mu_m n^{i\ell h/H}) \\ &= \frac{\mathbb{1}_{n \in \mathcal{P}}}{2n} (\lambda_m^2 + \mu_m^2 + 2\lambda_m \mu_m \cos((k - \ell)(\log n)h/H)). \end{aligned}$$

Moreover, for  $m \geq 1$ ,

$$n \geq e^{e^{K-1} \log H},$$

which implies

$$\begin{aligned} \left| \lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H}) \right| &\leq \frac{|\lambda_m| + |\mu_m|}{\sqrt{n}} \leq 2(\log T)^{1/20K} n^{-0.49} e^{-\frac{1}{100} e^{K-1} \log H} \\ &\ll n^{-0.49} (\log T)^{1/20K} e^{-\frac{1}{100} (\log T)^{(1-\delta)/K}} \\ &\leq n^{-0.49}, \end{aligned}$$

for  $T$  large enough depending on  $\delta$  and  $K$ . We deduce, by expanding the exponential:

$$\begin{aligned} & \mathbb{E} [e^{\lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H})}] \\ &= 1 + \frac{\mathbb{1}_{n \in \mathcal{P}}}{4n} (\lambda_m^2 + \mu_m^2 + 2\lambda_m \mu_m \cos((k - \ell)(\log n)h/H)) + \mathcal{O} \left( \sum_{r \geq 3} \frac{n^{-0.49r}}{r!} \right) \\ &= 1 + \frac{\mathbb{1}_{n \in \mathcal{P}}}{4n} (\lambda_m^2 + \mu_m^2 + 2\lambda_m \mu_m \cos((k - \ell)(\log n)h/H)) + \mathcal{O}(n^{-1.47}) \end{aligned}$$

For  $T$  large enough depending on  $\delta$  and  $K$ , the second term of the last formula is smaller than  $n^{-0.98}$ , which allows to take the logarithm and to use the estimate  $\log(1+x) = x + \mathcal{O}(x^2)$ , available for  $|x| \leq 1/2$ :

$$\begin{aligned} & \log \mathbb{E}[e^{\lambda_m \Re(Y_n n^{-ikh/H}) + \mu_m \Re(Y_n n^{-i\ell h/H})}] \\ &= \frac{\mathbb{1}_{n \in \mathcal{P}}}{4n} (\lambda_m^2 + \mu_m^2 + 2\lambda_m \mu_m \cos((k-\ell)(\log n)h/H)) + \mathcal{O}(n^{-1.47}), \end{aligned}$$

and then

$$\begin{aligned} & \mathbb{E}[\exp(\lambda_m V(k, m) + \mu_m V(\ell, m))] \\ &= \exp \left( E + \sum_{p \in \mathcal{P} \cap (e^{e^m \log H/K}, e^{e^{(m+1) \log H/K}]}] \frac{1}{4p} (\lambda_m^2 + \mu_m^2 + 2\lambda_m \mu_m \cos((k-\ell)(\log p)h/H)) \right), \end{aligned}$$

where

$$\begin{aligned} |E| &\ll \sum_{n \geq e^{e^{K-1} \log H}} n^{-1.47} \ll \exp(-0.47 H^{1/K}) \ll \exp(-0.47 (\log T)^{(1-\delta)/K}) \\ &\ll \exp(-(\log T)^{1/2K}). \end{aligned}$$

(recall that  $\delta \in (0, 1/2)$ ). Hence,

$$\begin{aligned} & \mathbb{E}[\exp(\lambda_m V(k, m) + \mu_m V(\ell, m))] \\ &= \exp \left( \left( S(e^{e^{(m+1) \log H/K}}) - S(e^{e^m \log H/K}) \right) \left( \frac{\lambda_m^2 + \mu_m^2}{4} \right) \dots \right. \\ & \quad \dots + \left( S(e^{e^{(m+1) \log H/K}}, (k-\ell)h/H) - S(e^{e^m \log H/K}, (k-\ell)h/H) \right) \left( \frac{\lambda_m \mu_m}{2} \right) \dots \\ & \quad \left. \dots + \mathcal{O} \left( \exp(-(\log T)^{1/2K}) \right) \right), \end{aligned}$$

where

$$S(x) := \sum_{p \in \mathcal{P} \cap [2, x]} \frac{1}{p}, \quad S(x, \theta) := \sum_{p \in \mathcal{P} \cap [2, x]} \frac{\cos(\theta \log p)}{p}.$$

For  $2 \leq x < y$ , let us write

$$I(x, y, \theta) := \int_x^y \frac{\cos(\theta \log t)}{t \log t} dt,$$

$$I(x, y) := I(x, y, 0) = \log \log y - \log \log x.$$

Integrating by parts, we get:

$$\begin{aligned} S(y, \theta) - S(x, \theta) &= \int_{(x, y]} \frac{\partial}{\partial t} I(x, t, \theta) d\rho(t) \\ &= I(x, y, \theta) + \int_{(x, y]} \frac{\partial}{\partial t} I(x, t, \theta) d(\rho(t) - t) \\ &= I(x, y, \theta) + \left[ \frac{\partial}{\partial t} I(x, t, \theta) (\rho(t) - t) \right]_{x+}^{y+} - \int_x^y \frac{\partial^2}{\partial t^2} I(x, t, \theta) (\rho(t) - t) dt, \end{aligned}$$

where, for  $t \in (x, y]$ ,

$$\rho(t) := \sum_{p \in \mathcal{P} \cap [2, t]} \log p = t + \mathcal{O} \left( t \exp(-a\sqrt{\log t}) \right) = t + \mathcal{O} \left( t \exp(-a\sqrt{\log x}) \right),$$

by a classical refinement of the prime number theorem due to de la Vallée Poussin ( $a > 0$  is a universal constant). Note that since we assume Riemann hypothesis, we could have used a better estimate of  $\rho(t)$ , but this is not needed for our computations. Now, we have

$$\left| \frac{\partial}{\partial t} I(x, t, \theta) \right| \leq \frac{1}{t \log t} \ll \frac{1}{t}$$

and

$$\begin{aligned} & \left| \frac{\partial^2}{\partial t^2} I(x, t, \theta) \right| \\ &= \left| \frac{-(\theta/t) \sin(\theta \log t)}{t \log t} - \frac{(1 + \log t) \cos(\theta \log t)}{t^2 \log^2 t} \right| \\ &\ll \frac{(1 + \theta)}{t^2 \log t} \end{aligned}$$

We then get, by estimating the bracket and the last integral,

$$S(y, \theta) - S(x, \theta) = I(x, y, \theta) + \mathcal{O} \left( (1 + \theta)(1 + \log \log y - \log \log x) \exp \left( -a \sqrt{\log x} \right) \right).$$

For  $x = e^{e^m \log H/K}$ ,  $y = e^{e^{(m+1)} \log H/K}$ ,  $\theta = 0$  or  $\theta = (k - \ell)h/H \in [-h, h]$ , the last error term is

$$\begin{aligned} &\ll_h \left( 1 + \frac{\log H}{K} \right) \exp \left( -a H^{1/2K} \right) \\ &\ll (1 + \log \log T) \exp(-a(\log T)^{(1-\delta)/2K}) \\ &\ll \exp(-(\log T)^{1/4K}), \end{aligned}$$

if  $T$  is large enough depending on  $\delta$  and  $K$ . Multiplying these error terms by  $(\lambda_m^2 + \mu_m^2)/4$  and  $\lambda_m \mu_m/2$  respectively, which are, in absolute value, at most  $(\log T)^{1/10K}$ , and adding them, we get something which is

$$\ll_h (\log T)^{1/10K} \exp(-(\log T)^{1/4K}) \ll \exp(-(\log T)^{1/5K})$$

for  $T$  large enough depending on  $\delta$  and  $K$ . Hence, we deduce

$$\begin{aligned} &\mathbb{E} [\exp (\lambda_m V(k, m) + \mu_m V(\ell, m))] \\ &= \exp \left( I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}) \left( \frac{\lambda_m^2 + \mu_m^2}{4} \right) \dots \right. \\ &\quad \dots + I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}, (k - \ell)h/H) \left( \frac{\lambda_m \mu_m}{2} \right) \dots \\ &\quad \left. \dots + \mathcal{O}_h \left( \exp \left( -(\log T)^{1/5K} \right) \right) \right). \end{aligned}$$

Using the independence of the variables  $V(k, m)$  for different values of  $m$  gives the following:

$$\mathbb{E} \left[ \exp \left( \sum_{m=1}^{K-1} (\lambda_m V(k, m) + \mu_m V(\ell, m)) \right) \right] = \exp \left( \mathcal{A} + \mathcal{O}_h \left( K \exp(-(\log T)^{1/5K}) \right) \right),$$

where

$$\mathcal{A} = \sum_{m=1}^{K-1} I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}) \left( \frac{\lambda_m^2 + \mu_m^2}{4} \right)$$



$$+ \sum_{m=1}^{K-1} I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}, (k-\ell)h/H) \left( \frac{\lambda_m \mu_m}{2} \right).$$

Since  $|I(x, y, \theta)| \leq \log \log y - \log \log x$  and  $\lambda_m, \mu_m \leq (\log T)^{1/20K}$ , we can bound  $\mathcal{A}$  by a telescopic sum which gives

$$|\mathcal{A}| \leq (\log T)^{1/10K} \log H \leq (\log T)^{1/10K} \log \log T,$$

and then

$$\begin{aligned} \mathbb{E} \left[ \exp \left( \sum_{m=1}^{K-1} (\lambda_m V(k, m) + \mu_m V(\ell, m)) \right) \right] &= \exp(\mathcal{A}) \left( 1 + \mathcal{O}_h \left( K \exp(-(\log T)^{1/5K}) \right) \right) \\ &= \exp(\mathcal{A}) + \mathcal{O}_h \left( K \exp \left( (\log T)^{1/10K} \log \log T - (\log T)^{1/5K} \right) \right) \\ &= \exp(\mathcal{A}) + \mathcal{O}_h \left( \exp(-(\log T)^{1/10K}) \right), \end{aligned}$$

for  $T$  large enough depending on  $K$  and  $\delta$ . This completes the proof of the proposition, provided that we check that

$$\mathbb{E} \left[ \exp \left( \sum_{m=1}^{K-1} (\lambda_m G(k, m) + \mu_m G(\ell, m)) \right) \right] = \exp(\mathcal{A}).$$

By independence of  $G(k, m)$  for different values of  $m$ , it is sufficient to check

$$\begin{aligned} &\mathbb{E} [\exp (\lambda_m G(k, m) + \mu_m G(\ell, m))] \\ &= \exp \left( I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}) \left( \frac{\lambda_m^2 + \mu_m^2}{4} \right) \dots \right. \\ &\quad \left. \dots + I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}, (k-\ell)h/H) \left( \frac{\lambda_m \mu_m}{2} \right) \right). \end{aligned}$$

Since  $(G(k, m), G(\ell, m))$  is a centered Gaussian vector, it is sufficient to check that its covariance structure is given by

$$\begin{aligned} \mathbb{E}[(G(k, m))^2] &= \mathbb{E}[(G(\ell, m))^2] = \frac{1}{2} I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}), \\ \mathbb{E}[G(k, m)G(\ell, m)] &= \frac{1}{2} I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}, (k-\ell)h/H). \end{aligned}$$

By including the case  $k = \ell$ , it is enough to check the last equality, which is proven as follows, using the fact that (formally)  $\mathbb{E}[(dW_t)^2] = 0$  and  $\mathbb{E}[dW_t \overline{dW_t}] = 2dt$ ,

$$\begin{aligned} &\mathbb{E}[G(k, m)G(\ell, m)] \\ &= \mathbb{E} \left[ \left( \Re \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \frac{t^{-ikh/H}}{\sqrt{2t \log t}} dW_t \right) \left( \Re \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \frac{t^{-i\ell h/H}}{\sqrt{2t \log t}} dW_t \right) \right] \\ &\quad \frac{1}{4} \left[ \left( \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \left( \frac{t^{-ikh/H}}{\sqrt{2t \log t}} dW_t + \frac{t^{ikh/H}}{\sqrt{2t \log t}} \overline{dW_t} \right) \dots \right. \right. \\ &\quad \left. \left. \dots \times \left( \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \left( \frac{t^{-i\ell h/H}}{\sqrt{2t \log t}} dW_t + \frac{t^{i\ell h/H}}{\sqrt{2t \log t}} \overline{dW_t} \right) \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \frac{2t^{-ikh/H} t^{i\ell h/H} + 2t^{ikh/H} t^{-i\ell h/H}}{2t \log t} dt \\
&= \frac{1}{4} \int_{e^{e^m \log H/K}}^{e^{e^{(m+1)} \log H/K}} \frac{4 \cos((k-\ell)h(\log t)/H)}{2t \log t} dt \\
&= \frac{1}{2} I(e^{e^m \log H/K}, e^{e^{(m+1)} \log H/K}, (k-\ell)h/H).
\end{aligned}$$

□

Combining Propositions 6.1 and 6.2, we obtain that the Fourier transforms of the variables  $S_0(k, m)$  and the Gaussian variables  $G(k, m)$  are close to each other. By using Fourier inversion, we deduce the following, which is the main result of the section:

**Proposition 6.4.** *Let  $\varphi$  be a smooth function with compact support,  $k, \ell \in \{0, \dots, H-1\}$ ,  $x \in \mathbb{R}$ , and  $A > 0$ . Then, with the notation of the previous propositions:*

$$\mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \right] = \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(G(k, m) - x) \right] + \mathcal{O}_{\varphi, K, \delta, h, A}((\log T)^{-A})$$

and

$$\begin{aligned}
\mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \varphi(S_0(\ell, m) - x) \right] &= \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(G(k, m) - x) \varphi(G(\ell, m) - x) \right] \\
&\quad + \mathcal{O}_{\varphi, K, \delta, h, A}((\log T)^{-A})
\end{aligned}$$

*Proof.* Let us prove the second estimate: the proof of the first one is exactly similar. Let  $\eta$  be the inverse Fourier transform of  $\varphi$ : from the assumptions of  $\varphi$ ,  $\eta$  is bounded and dominated by any negative power at infinity. We have

$$\varphi(S_0(k, m) - x) = \int_{-\infty}^{\infty} \eta(\lambda) e^{-i\lambda(S_0(k, m) - x)} d\lambda$$

and then

$$\begin{aligned}
&\prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \varphi(S_0(\ell, m) - x) \\
&= \int_{\mathbb{R}^{2K-2}} \prod_{m=1}^{K-1} \eta(\lambda_m) \eta(\mu_m) e^{ix \sum_{m=1}^{K-1} (\lambda_m + \mu_m)} e^{-i \sum_{m=1}^{K-1} (\lambda_m S_0(k, m) + \mu_m S_0(\ell, m))} \prod_{m=1}^{K-1} d\lambda_m d\mu_m,
\end{aligned}$$

and a similar formula for the variables  $G(k, m)$ ,  $G(\ell, m)$ . Therefore,

$$\begin{aligned}
&\left| \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \varphi(S_0(\ell, m) - x) \right] - \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(G(k, m) - x) \varphi(G(\ell, m) - x) \right] \right| \\
&\leq \int_{\mathbb{R}^{2K-2}} \prod_{m=1}^{K-1} |\eta(\lambda_m)| |\eta(\mu_m)| \dots \\
&\dots \times \left| \mathbb{E} [e^{-i \sum_{m=1}^{K-1} (\lambda_m S_0(k, m) + \mu_m S_0(\ell, m))}] - \mathbb{E} [e^{-i \sum_{m=1}^{K-1} (\lambda_m G(k, m) + \mu_m G(\ell, m))}] \right| \prod_{m=1}^{K-1} d\lambda_m d\mu_m.
\end{aligned}$$

If  $T$  is large enough depending only on  $K$  and  $\delta$ , we deduce, from the two previous propositions:

$$\left| \mathbb{E}[e^{-i \sum_{m=1}^{K-1} (\lambda_m S_0(k, m) + \mu_m S_0(\ell, m))}] - \mathbb{E}[e^{-i \sum_{m=1}^{K-1} (\lambda_m G(k, m) + \mu_m G(\ell, m))}] \right| \ll_h \exp(-(\log T)^{\delta/5K}),$$

if  $|\lambda_m|, |\mu_m| \leq (\log T)^{\delta/50K}$  for  $m \in \{1, \dots, K-1\}$ . Since the difference of expectations is unconditionally bounded by 2, we get, for  $T$  large enough depending on  $\delta$  and  $K$ :

$$\begin{aligned} & \left| \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \varphi(S_0(\ell, m) - x) \right] - \mathbb{E} \left[ \prod_{m=1}^{K-1} \varphi(G(k, m) - x) \varphi(G(\ell, m) - x) \right] \right| \\ & \ll_h (||\eta||_\infty)^{2K-2} \left( [2(\log T)^{\delta/50K}]^{2K-2} \exp(-(\log T)^{\delta/5K}) \right) \\ & + \int_{\mathbb{R}^{2K-2}} \prod_{m=1}^{K-1} |\eta(\lambda_m)| |\eta(\mu_m)| \mathbb{1}_{\exists m \in \{1, \dots, K-1\}, \max(|\lambda_m|, |\mu_m|) \geq (\log T)^{\delta/50K}} d\lambda_m d\mu_m. \end{aligned}$$

It is immediate that the first term is  $\mathcal{O}_{\eta, \delta, K, A}((\log T)^{-A})$ . By bounding the indicator of a union by the sum of the indicators, we see that the second term is at most:

$$(2K-2)(||\eta||_1)^{2K-3} \int_{\mathbb{R} \setminus (-\log T)^{\delta/50K}, (\log T)^{\delta/50K}} |\eta(\lambda)| d\lambda.$$

Since the tail of  $\eta$  decays faster than any power, we see that the second term is also  $\mathcal{O}_{\eta, \delta, K, A}((\log T)^{-A})$ . This proves the proposition for  $T$  large enough depending on  $\delta$  and  $K$ : we can then remove this assumption by increasing the implicit constant in  $\mathcal{O}_{\varphi, K, \delta, h, A}$ .  $\square$

## 7 The lower bound in the main theorem

We start this section by proving an estimate of the expectations involved in Proposition 6.4. This will give some information on our problem, by using the following remark: if  $x > 0$ , if  $\varphi$  vanishes on  $\mathbb{R}_-$  and if

$$\prod_{m=1}^{K-1} \varphi(S_0(k, m) - x) \neq 0,$$

then  $S_0(k, m) > x$  for all  $m \in \{1, \dots, K-1\}$ , and then

$$\sum_{m=1}^{K-1} S(k, m) \geq \sum_{m=1}^{K-1} S_0(k, m) \geq (K-1)x.$$

We first consider the right-hand sides involved in Proposition 6.4, which only depend on Gaussian variables. We can bound the covariance of  $G(k, m)$  and  $G(\ell, m)$  as follows:

**Lemma 7.1.** *We have:*

$$\mathbb{E}[G(k, m)^2] = \frac{\log H}{2K}$$

and

$$|\mathbb{E}[G(k, m)G(\ell, m)]| \ll_h \frac{H^{1-(m/K)}}{|k - \ell|}.$$

*Proof.* We have seen that

$$\mathbb{E}[G(k, m)G(\ell, m)] = \frac{1}{2}I(x, y, \theta),$$

where

$$x = e^{e^m \log H/K}, y = e^{e^{(m+1)} \log H/K}, \theta = (k - \ell)h/H,$$

and

$$I(x, y, \theta) = \int_x^y \frac{\cos(\theta \log t)}{t \log t} dt = \int_{\log x}^{\log y} \frac{\cos(\theta u)}{u} du.$$

For  $k = \ell$  and then  $\theta = 0$ , we deduce

$$\mathbb{E}[G(k, m)^2] = \frac{1}{2}(\log \log y - \log \log x) = \frac{\log H}{2K}.$$

For  $k \neq \ell$  and then  $\theta \neq 0$ , we have

$$I(x, y, \theta) = \int_{|\theta| \log x}^{|\theta| \log y} \frac{\cos v}{v} dv \ll \frac{1}{|\theta| \log x}$$

by integration by parts, which gives

$$|\mathbb{E}[G(k, m)G(\ell, m)]| \ll \frac{H}{h|k - \ell|} e^{-m \log H/K}$$

and the conclusion of the proposition.  $\square$

This lemma is used to get the following estimates of the quantities involved in Proposition 6.4:

**Proposition 7.2.** *Let us take the notation above, and let us assume that  $\varphi$  is not identically zero, smooth, with compact support, nonnegative and equal to zero on  $\mathbb{R}_-$ . Then, for  $k, \ell \in \{0, 1, \dots, H - 1\}$ ,  $m \in \{1, \dots, K - 1\}$ ,  $\nu \in (0, 1)$ ,  $x = K^{-1}(\log H)\sqrt{1 - \nu}$ , we have*

$$\frac{H^{-(1-\nu)/K}}{\sqrt{\log H}} \ll_{K, \varphi} \mathbb{E}[\varphi(G(k, m) - x)] \ll_{K, \varphi} \frac{H^{-(1-\nu)/K}}{\sqrt{\log H}},$$

for  $|k - \ell| > H^{1-(m/K)}$ ,

$$\mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \ll_{h, K, \varphi} H^{-2(1-\nu)/K}$$

and for  $|k - \ell| > H^{1-(1/2K)}$ ,

$$\begin{aligned} & \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \\ &= \mathbb{E}[\varphi(G(k, m) - x)]\mathbb{E}[\varphi(G(\ell, m) - x)] \left(1 + \mathcal{O}_{h, K, \varphi}(H^{-1/2K})\right). \end{aligned}$$

*Proof.* We have

$$\mathbb{E}[\varphi(G(k, m) - x)] = \frac{1}{\sqrt{2\pi(\log H)/2K}} \int_0^A e^{-K(x+t)^2/(\log H)} \varphi(t) dt,$$

if the support of  $\varphi$  is included in  $[0, A]$ . Hence,

$$\mathbb{E}[\varphi(G(k, m) - x)] \leq \frac{A\|\varphi\|_\infty}{\sqrt{2\pi(\log H)/2K}} e^{-Kx^2/(\log H)} \ll_{K, \varphi} \frac{H^{-(1-\nu)/K}}{\sqrt{\log H}}.$$

On the other hand, if  $\varphi$  is larger than  $\varepsilon > 0$  on an interval  $[t_0, t_1] \subset [0, A]$ ,

$$\mathbb{E}[\varphi(G(k, m) - x)] \geq \frac{\varepsilon(t_1 - t_0)}{\sqrt{2\pi(\log H)/2K}} e^{-K(x+t_1)^2/(\log H)} \gg_{K, \varphi} \frac{H^{-(1-\nu)/K}}{\sqrt{\log H}},$$

since

$$\frac{K[(x+t_1)^2 - x^2]}{\log H} = \frac{Kt_1(2x+t_1)}{\log H} \ll_{K,\varphi} 1.$$

For the second estimate, we observe that

$$\mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \leq \|\varphi\|_\infty^2 \mathbb{P}[G(k, m), G(\ell, m) \in [x, x + A]].$$

Now, we can write the equality in distribution

$$G(k, m) = \rho G(\ell, m) + \tilde{G}\sqrt{1 - \rho^2},$$

where  $\tilde{G}$  is independent of  $G(\ell, m)$ , with the same variance, and

$$\rho := \frac{\mathbb{E}[G(k, m)G(\ell, m)]}{\mathbb{E}[G(\ell, m)]^2}.$$

If  $|k - \ell| > H^{1-(m/K)}$ , then

$$|\rho| \ll_h \frac{H^{1-(m/K)}}{|k - \ell|(\log H)/2K} \ll_{h,K} \frac{1}{\log H}$$

We deduce that if  $G(k, m)$  and  $G(\ell, m)$  are in  $[x, x + A]$ , and if  $H$  is large enough depending on  $h$  and  $K$ ,

$$\begin{aligned} \tilde{G} &= \frac{1}{\sqrt{1 - \rho^2}} (G(k, m) - \rho G(\ell, m)) \\ &= \left(1 - \mathcal{O}_{h,K} \left(\frac{1}{\log^2 H}\right)\right)^{-1/2} \left(x + \mathcal{O}(A) - \mathcal{O}_{h,K} \left(\frac{x + A}{\log H}\right)\right) \\ &= x + \mathcal{O}_{h,K,\varphi}(1). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] &\ll_\varphi \mathbb{P}[G(\ell, m), \tilde{G} \geq x - \mathcal{O}_{h,K,\varphi}(1)] \\ &\leq \exp\left(-\frac{2K(x - \mathcal{O}_{h,K,\varphi}(1))^2_+}{\log H}\right) \ll_{h,K,\varphi} \exp\left(-\frac{2Kx^2}{\log H}\right) = H^{-2(1-\nu)/K}, \end{aligned}$$

for  $H$  large enough depending on  $h$  and  $K$ . This condition can then be removed by changing the implicit constant of the estimate.

For the last estimate, under the assumption  $|k - \ell| > H^{1-(1/2K)}$ , we have, as soon as  $C$  is invertible:

$$\begin{aligned} &\mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] - \mathbb{E}[\varphi(G(k, m) - x)]\mathbb{E}[\varphi(G(\ell, m) - x)] \\ &= \frac{1}{2\pi} \int_0^A \int_0^A \left( \frac{e^{-\frac{1}{2}(t+x, u+x)C^{-1}(t+x, u+x)^t}}{\sqrt{\det(C)}} - \frac{e^{-\frac{1}{2}(t+x, u+x)C_0^{-1}(t+x, u+x)^t}}{\sqrt{\det(C_0)}} \right) \varphi(t)\varphi(u) dt du, \end{aligned}$$

where  $C$  is the covariance matrix of  $(G(k, m), G(\ell, m))$ , and  $C_0$  its diagonal part. Now,

$$|\mathbb{E}[G(k, m)G(\ell, m)]| \ll_h \frac{H^{1-(m/K)}}{|k - \ell|} \leq \frac{H^{1-(1/K)}}{H^{1-(1/2K)}} \leq H^{-1/2K},$$

$$C_0 = \begin{pmatrix} \frac{\log H}{2K} & 0 \\ 0 & \frac{\log H}{2K} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{\log H}{2K} & \mathcal{O}_h(H^{-1/2K}) \\ \mathcal{O}_h(H^{-1/2K}) & \frac{\log H}{2K} \end{pmatrix},$$

$$\det(C_0) = \frac{\log^2 H}{4K^2},$$

$$\det(C) = \frac{\log^2 H}{4K^2} + \mathcal{O}_h(H^{-1/K}).$$

For  $H$  large enough depending only on  $h$  and  $K$ ,  $C$  is invertible and

$$\begin{aligned} C^{-1} &= \left( \frac{\log^2 H}{4K^2} + \mathcal{O}_h(H^{-1/K}) \right)^{-1} \begin{pmatrix} \frac{\log H}{2K} & \mathcal{O}_h(H^{-1/2K}) \\ \mathcal{O}_h(H^{-1/2K}) & \frac{\log H}{2K} \end{pmatrix} \\ &= \left( 1 + \mathcal{O}_{h,K}(H^{-1/K} \log^{-2} H) \right) \begin{pmatrix} \frac{2K}{\log H} & \mathcal{O}_{h,K}(H^{-1/2K} \log^{-2} H) \\ \mathcal{O}_{h,K}(H^{-1/2K} \log^{-2} H) & \frac{2K}{\log H} \end{pmatrix} \\ &= C_0^{-1} + \mathcal{O}_{h,K}(H^{-1/2K} \log^{-2} H). \end{aligned}$$

We then have

$$|(t+x, u+x)(C^{-1} - C_0^{-1})(t+x, u+x)^t| \ll_{h,K,\varphi} (\log H)(H^{-1/2K} \log^{-2} H)(\log H) = H^{-1/2K}$$

whereas

$$\sqrt{\frac{\det(C)}{\det(C_0)}} = 1 + \mathcal{O}_{h,K}(H^{-1/K} \log^{-2} H),$$

which implies

$$\frac{e^{-\frac{1}{2}(t+x, u+x)C^{-1}(t+x, u+x)^t}}{\sqrt{\det(C)}} = \frac{e^{-\frac{1}{2}(t+x, u+x)C_0^{-1}(t+x, u+x)^t}}{\sqrt{\det(C_0)}} \left( 1 + \mathcal{O}_{h,K,\varphi}(H^{-1/2K}) \right),$$

$$\begin{aligned} &\mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] - \mathbb{E}[\varphi(G(k, m) - x)]\mathbb{E}[\varphi(G(\ell, m) - x)] \\ &= \frac{1}{2\pi} \int_0^A \int_0^A \frac{e^{-\frac{1}{2}(t+x, u+x)C_0^{-1}(t+x, u+x)^t}}{\sqrt{\det(C_0)}} \mathcal{O}_{h,K,\varphi}(H^{-1/2K}) \varphi(t) \varphi(u) dt du \\ &\ll_{h,K,\varphi} \frac{H^{-1/2K}}{2\pi} \int_0^A \int_0^A \frac{e^{-\frac{1}{2}(t+x, u+x)C_0^{-1}(t+x, u+x)^t}}{\sqrt{\det(C_0)}} \varphi(t) \varphi(u) dt du \end{aligned}$$

by positivity of the last integrand, which proves the result of the proposition, for  $H$  large enough depending on  $h$  and  $K$ . Again, this assumption can be removed by changing the implicit constant in  $\mathcal{O}_{h,K,\varphi}$ .  $\square$

By using Propositions 6.4 and 7.2, we can apply the second moment method to the random variables  $S_0(k, m)$ :

**Proposition 7.3.** *With the notation and under the assumptions of the two previous propositions, we have, for  $T$  large enough depending on  $\varphi, K, \delta, h$ , and for  $\nu \in (0, 1/2)$ ,*

$$\mathbb{E}[J^2] = (\mathbb{E}[J])^2 (1 + \mathcal{O}_{\varphi,K,\delta,h}((\log T)^{-\nu(1-\delta)/K} (\log \log T)^{K-1})),$$

where

$$J := \sum_{k=0}^{H-1} \prod_{m=1}^{K-1} \varphi(S_0(k, m) - x).$$

*Proof.* Using Proposition 6.4 and the independence of the Gaussian variables  $G(k, m)$ , we get

$$\begin{aligned}\mathbb{E}[J] &= \sum_{k=0}^{H-1} \left( \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)] + \mathcal{O}_{\varphi, K, \delta, h, A}((\log T)^{-A}) \right) \\ &= H (\mathbb{E}[\varphi(G(0, 1) - x)])^{K-1} + \mathcal{O}_{\varphi, K, \delta, h, B}(H^{-B})\end{aligned}$$

for all  $B > 0$ , since  $H \leq \log T$ . Hence, by taking  $B = 1$  (say), and by using Proposition 7.2,

$$\mathbb{E}[J] \gg_{\varphi, K, \delta, h} \frac{H^{1-(1-\nu)(K-1)/K}}{(\log H)^{(K-1)/2}}$$

for  $H$ , and then  $T$ , large enough depending on  $\varphi, K, \delta, h$ . On the other hand,

$$\begin{aligned}\mathbb{E}[J^2] &= \sum_{0 \leq k, \ell \leq H-1} \left( \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] + \mathcal{O}_{\varphi, K, \delta, h, A}((\log T)^{-A}) \right) \\ &= \sum_{0 \leq k, \ell \leq H-1} \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] + \mathcal{O}_{\varphi, K, \delta, h, B}(H^{-B}).\end{aligned}$$

For  $H^{1-(r/K)} < |k - \ell| \leq H^{1-((r-1)/K)}$ ,  $r$  integer between 1 and  $K$ , or  $|k - \ell| \leq 1$  for  $r = K$ , we get

$$\begin{aligned}\prod_{m=1}^{r-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] &\leq \|\varphi\|_{\infty}^{r-1} \prod_{m=1}^{r-1} \mathbb{E}[\varphi(G(k, m) - x)] \\ &\ll_{K, \varphi} \left( \frac{H^{-(1-\nu)/K}}{\sqrt{\log H}} \right)^{r-1} \ll_{K, \varphi} H^{-(r-1)(1-\nu)/K},\end{aligned}$$

and from the fact that  $|k - \ell| > H^{1-(m/K)}$  for  $m \geq r$  (which implies that we do not have  $r = K$ ),

$$\prod_{m=r}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \ll_{h, K, \varphi} H^{-2(K-r)(1-\nu)/K},$$

and then

$$\prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \ll_{h, K, \varphi} H^{-[r-1+2(K-r)](1-\nu)/K}.$$

For  $2 \leq r \leq K$ , the number of couples  $(k, \ell)$  with  $|k - \ell| \leq H^{1-((r-1)/K)}$  is dominated by  $H^{2-((r-1)/K)}$ , which gives

$$\begin{aligned}\sum_{0 \leq k, \ell \leq H-1, H^{1-(r/K)} < |k-\ell| \leq H^{1-((r-1)/K)}} \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)\varphi(G(\ell, m) - x)] \\ \ll_{h, K, \varphi} H^{2-[r-1+(1-\nu)(r-1+2(K-r))]/K},\end{aligned}$$

where

$$\begin{aligned}2 - \frac{r-1+(1-\nu)(r-1+2(K-r))}{K} \\ = \frac{2K+1+(1-\nu)-2K(1-\nu)}{K} - \frac{r}{K} (1+(1-\nu)-2(1-\nu))\end{aligned}$$

$$= \frac{2K\nu + 2 - \nu}{K} - \frac{r\nu}{K} \leq 2\nu + \frac{2 - 3\nu}{K},$$

since  $r \geq 2$ . Hence,

$$\sum_{0 \leq k, \ell \leq H-1, |k-\ell| \leq H^{1-(1/K)}} \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x) \varphi(G(\ell, m) - x)] \\ \ll_{h, K, \varphi} H^{2\nu + (2-3\nu)/K}.$$

The number of couples  $(k, \ell)$  with  $|k - \ell| \leq H^{1-(1/2K)}$  is dominated by  $H^{2-(1/2K)}$ , and then, by using the case  $r = 1$ ,

$$\sum_{0 \leq k, \ell \leq H-1, H^{1-(1/K)} < |k-\ell| \leq H^{1-(1/2K)}} \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x) \varphi(G(\ell, m) - x)] \\ \ll_{h, K, \varphi} H^{2-(1/2K) - (2(1-\nu)(K-1)/K)},$$

with

$$2 - \frac{1}{2K} - \frac{2(1-\nu)(K-1)}{K} \\ = \frac{2K - (1/2) - 2K(1-\nu) + 2(1-\nu)}{K} \\ = \frac{2K\nu + (3/2) - 2\nu}{K} = 2\nu + \frac{(3/2) - 2\nu}{K} \leq 2\nu + \frac{2 - 3\nu}{K},$$

since  $\nu < 1/2$ , and then

$$\sum_{0 \leq k, \ell \leq H-1, |k-\ell| \leq H^{1-(1/2K)}} \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x) \varphi(G(\ell, m) - x)] \\ \ll_{h, K, \varphi} H^{2\nu + (2-3\nu)/K}.$$

By using the last estimate of Proposition 7.2, available for  $|k - \ell| > H^{1-(1/2K)}$ , we deduce

$$\mathbb{E}[J^2] \leq \sum_{0 \leq k, \ell \leq H-1} \left( \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)] \mathbb{E}[\varphi(G(\ell, m) - x)] \left( 1 + \mathcal{O}_{h, K, \varphi}(H^{-1/2K}) \right) \right) \\ + \mathcal{O}_{h, K, \varphi}(H^{2\nu + (2-3\nu)/K}) + \mathcal{O}_{\varphi, K, \delta, h, B}(H^{-B}) \\ = \left( 1 + \mathcal{O}_{h, K, \varphi}(H^{-1/2K}) \right) \sum_{0 \leq k, \ell \leq H-1} \left( \prod_{m=1}^{K-1} \mathbb{E}[\varphi(G(k, m) - x)] \mathbb{E}[\varphi(G(\ell, m) - x)] \right) \\ + \mathcal{O}_{\varphi, K, \delta, h}(H^{2\nu + (2-3\nu)/K}) \\ = \left( 1 + \mathcal{O}_{h, K, \varphi}(H^{-1/2K}) \right) (H(\mathbb{E}[\varphi(G(0, 1) - x)])^{K-1})^2 + \mathcal{O}_{\varphi, K, \delta, h}(H^{2\nu + (2-3\nu)/K}) \\ = \left( 1 + \mathcal{O}_{h, K, \varphi}(H^{-1/2K}) \right) (\mathbb{E}[J] + \mathcal{O}_{\varphi, K, \delta, h, B}(H^{-B}))^2 + \mathcal{O}_{\varphi, K, \delta, h}(H^{2\nu + (2-3\nu)/K}).$$

Expanding the square and using the trivial estimate

$$\mathbb{E}[J] \leq H \|\varphi\|_{\infty}^{K-1} \ll_{K, \varphi} H,$$

we get

$$(\mathbb{E}[J] + \mathcal{O}_{\varphi, K, \delta, h, B}(H^{-B}))^2 = (\mathbb{E}[J])^2 + \mathcal{O}_{\varphi, K, \delta, h, B'}(H^{-B'}),$$



and then

$$\mathbb{E}[J^2] \leq \left(1 + \mathcal{O}_{h,K,\varphi}(H^{-1/2K})\right) (\mathbb{E}[J])^2 + \mathcal{O}_{\varphi,K,\delta,h}(H^{2\nu+(2-3\nu)/K}).$$

Now, for  $T$  large enough depending on  $\varphi, K, \delta, h$ ,

$$(\mathbb{E}[J])^2 \gg_{\varphi,K,\delta,h} (\log H)^{-(K-1)} H^{2-(2-2\nu)(K-1)/K},$$

where

$$\begin{aligned} 2 - \frac{(2-2\nu)(K-1)}{K} &= \frac{2K - 2K + 2 + 2K\nu - 2\nu}{K} \\ &= 2\nu + \frac{2-2\nu}{K}, \end{aligned}$$

and then

$$\mathbb{E}[J^2] \leq (\mathbb{E}[J])^2 (1 + \mathcal{O}_{\varphi,K,\delta,h}(H^{-\nu/K} (\log H)^{K-1})),$$

(recall that  $1/2K \geq \nu/K$ ), the converse inequality being obvious.  $\square$

Using this bound on the second moment and the Paley-Zygmund inequality, we deduce the following:

**Proposition 7.4.** *For all  $\omega < (1-\delta)(K-1)/K$ , we have, with probability tending to 1 when  $T$  goes to infinity:*

$$\sup_{k \in \{0, \dots, H-1\}} \Re \left( \kappa \sum_{p \in \mathcal{P} \cap (e^{e^{\log H/K}}, e^H]} p^{-\frac{1}{2}-i(TU-h/2+kh/H)} \right) \geq \omega \log \log T.$$

*Proof.* With the notation of the previous proposition, the Paley-Zygmund inequality can be written as follows, using the Cauchy-Schwarz inequality (recall that  $J \geq 0$ ):

$$\mathbb{P}[J > 0] \mathbb{E}[J^2] = (\mathbb{P}[J > 0])^2 \mathbb{E}[J^2 | J > 0] \geq (\mathbb{P}[J > 0])^2 (\mathbb{E}[J | J > 0])^2 = (\mathbb{E}[J])^2,$$

$$\mathbb{P}[J > 0] \geq \frac{(\mathbb{E}[J])^2}{\mathbb{E}[J^2]},$$

which, by the previous proposition, tends to 1 when  $T$  goes to infinity, for fixed  $\varphi, K, \delta, h, \nu$ . Hence, with probability tending to 1, there exists  $k \in \{0, 1, \dots, H-1\}$  such that for all  $m \in \{1, \dots, K-1\}$ ,  $\varphi(S_0(k, m) - x) > 0$ , which implies  $S_0(k, m) \geq x$ , and then  $S(k, m) \geq x$ , since  $S_0(k, m)$  is a truncation of  $S(k, m)$ . Summing this inequality for  $m \in \{1, \dots, K-1\}$ , we deduce

$$\begin{aligned} \Re \left( \kappa \sum_{p \in \mathcal{P} \cap (e^{e^{\log H/K}}, e^H]} p^{-\frac{1}{2}-i(TU-h/2+kh/H)} \right) &\geq (K-1)x = (K-1)(\log H) \sqrt{1-\nu}/K \\ &\sim_{T \rightarrow \infty} (K-1)(1-\delta)(\log \log T) \sqrt{1-\nu}/K. \end{aligned}$$

If we fix  $\nu \in (0, 1/2)$  small enough, the lower bound we obtain is larger than  $\omega \log \log T$  for  $T$  large enough.  $\square$

In the previous proposition, the sum on primes which is involved starts at  $p \geq e^{e^{\log H/K}}$ . In order to solve our main problem, it remains to control the sum over the primes smaller than  $e^{e^{\log H/K}}$ . This is given by the following proposition:

**Proposition 7.5.** *With probability tending to 1 when  $T$  goes to infinity:*

$$\inf_{k \in \{0, \dots, H-1\}} \Re \left( \kappa \sum_{p \in \mathcal{P} \cap [3, e^{\log H/K}]} p^{-\frac{1}{2} - i(TU - h/2 + kh/H)} \right) \geq -\frac{2}{\sqrt{K}} \log \log T.$$

*Proof.* By Proposition 6.1, we have, for  $|\lambda| \leq (\log T)^{\delta/100}$ ,  $k \in \{0, 1, \dots, H-1\}$ ,

$$\mathbb{E}[\exp(\lambda S_0(k, 0))] = \mathbb{E}[\exp(\lambda V(k, 0))] + \mathcal{O}\left(\exp(-(\log T)^{\delta/4})\right).$$

On the other hand,

$$\begin{aligned} \mathbb{E}[\exp(\lambda V(k, 0))] &= \prod_{p \in \mathcal{P} \cap [3, e^{\log H/K}]} \mathbb{E}[e^{\lambda \Re X_p / \sqrt{p}}] \\ &= \prod_{p \in \mathcal{P} \cap [3, e^{\log H/K}]} \mathbb{E}[\cosh(\lambda \Re X_p / \sqrt{p})] \\ &\leq \prod_{p \in \mathcal{P} \cap [3, e^{\log H/K}]} \cosh(\lambda / \sqrt{p}) \\ &\leq \exp \left( \frac{\lambda^2}{2} \sum_{p \in \mathcal{P} \cap [3, e^{\log H/K}]} \frac{1}{p} \right) \\ &= \exp \left( \frac{\lambda^2}{2} \left( \frac{\log H}{K} + \mathcal{O}(1) \right) \right). \end{aligned}$$

We deduce, for  $\lambda = -2\sqrt{K}$ , and  $T$  large enough depending on  $K$  and  $\delta$  (in order to have  $|\lambda| \leq (\log T)^{\delta/100}$ ),

$$\begin{aligned} \mathbb{E}[\exp(-2\sqrt{K} S_0(k, 0))] &\ll_K H^2, \\ \mathbb{P}[S_0(k, 0) \leq -2K^{-1/2} \log H] &\leq H^{-4} \mathbb{E}[\exp(-2\sqrt{K} S_0(k, 0))] \ll_K H^{-2}. \end{aligned}$$

By a union bound, we have with probability tending to 1,  $S_0(k, 0) \geq -2K^{-1/2} \log H$ , and then  $S(k, 0) \geq -2K^{-1/2} \log H$  for  $T$  large enough depending on  $K$  and  $\delta$ , since the truncation  $-(\log T)^{\delta/3}$  is strictly below  $-2K^{-1/2} \log H$ . This gives the desired result.  $\square$

We deduce the lower bound part of Theorem 1.2:

**Proposition 7.6.** *Let us assume the Riemann hypothesis. For  $h > 0$ ,  $U$  uniform on  $[0, 1]$ ,  $\epsilon \in (0, 1)$ ,  $\kappa \in \{1, -i, i\}$ , we have, with probability tending to 1 when  $T$  goes to infinity:*

$$\sup_{\tau \in [UT-h, UT+h]} \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \geq (1 - \epsilon) \log \log T.$$

*Proof.* By combining Propositions 5.1, 7.4 and 7.5, we have, with probability tending to 1,

$$\sup_{\tau \in [UT-h, UT+h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \geq \left( \omega - \frac{2}{\sqrt{K}} \right) \log \log T - \mathcal{O}_h(\sqrt{\log \log T}),$$

for all  $\omega < (1 - \delta)(K - 1)/K$ . If we choose  $\delta$  sufficiently small and  $K$  sufficiently large, depending on  $\epsilon$ , we deduce, with probability tending to 1,

$$\sup_{\tau \in [UT-h, UT+h]} \left( \Re \left( \kappa \log \zeta \left( \frac{1}{2} + i\tau \right) \right) \right)_+ \geq (1 - (\epsilon/2)) \log \log T - \mathcal{O}_h(\sqrt{\log \log T}) \geq (1 - \epsilon) \log \log T,$$

for  $T$  large enough depending on  $h$  and  $\epsilon$ . Since the lower bound is positive, we can then remove the positive part in  $\Re(\kappa \log \zeta)$ .  $\square$

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